

Hands-on Tutorial on Optimization

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Modeling tricks

Linear programming tricks

$$\begin{array}{ll} \min & \sum_{j \in J} c_j x_j \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \begin{array}{l} \leq \\ \geq \end{array} b_i & \forall i \\ & x_j \geq 0 & \forall j \end{array}$$

Absolute values

$$\begin{array}{ll} \min & \sum_{j \in J} c_j |x_j| \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \begin{array}{l} \leq \\ \geq \end{array} b_i \quad \forall i \end{array}$$

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Introduce non-negative variables x_j^+ , x_j^- .

$$\begin{array}{l} x_j = \\ |x_j| = \\ x_j^+, x_j^- \geq 0 \end{array}$$

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Absolute values

$$\begin{array}{ll} \min & \sum_{j \in J} c_j (x_j^+ + x_j^-) \\ \text{s.t.} & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \begin{array}{l} \leq \\ \geq \end{array} b_i & \forall i \\ & x_j^+, x_j^- \geq 0 & \forall j \end{array}$$

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Minimax objective

$$\begin{array}{ll} \min & \max_{k \in K} \sum_{j \in J} c_{kj} x_j \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \leq b_i \quad \forall i \end{array}$$

Minimax objective

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Introduce variable z :

$$z = \max_{k \in K} \sum_{j \in J} c_{kj} x_j \quad \forall k \in K$$

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Integer linear programming tricks

$$\begin{array}{ll} \min & \sum_{j \in J} c_j x_j \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \begin{array}{l} \leq \\ \geq \end{array} b_i & \forall i \\ & x_j \in \{0, 1\} / \mathbb{Z} & \forall j \end{array}$$

Jump in value

Constraint:

$$x = 0 \quad \text{or} \quad \ell \leq x \leq u$$

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Add binary variable y :

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } \ell \leq x \leq u \end{cases}$$

and constraints:

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Add binary variable y :

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } \ell \leq x \leq u \end{cases}$$

and constraints:

$$x \leq uy$$

$$x \geq \ell y$$

$$y \in \{0, 1\}$$

Fixed costs

$$\begin{array}{ll} \min & C(x) \\ \text{s.t.} & a_i x + \sum_{j \in J} a_{ij} w_j \leq b_i \quad \forall i \\ & x \geq 0 \\ & w_j \geq 0 \quad \forall j \end{array}$$

with

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

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and we have constraints:

$$\begin{array}{l} x \leq uy \\ y \in \{0, 1\} \end{array}$$

Either or constraints

$$\begin{array}{ll} \min & \sum_{j \in J} c_j x_j \\ \text{s.t.} & \sum_{j \in J} a_{1j} x_j \leq b_1 \quad \text{or} \quad \sum_{j \in J} a_{2j} x_j \leq b_2 \\ & x_j \geq 0 \quad \forall j \end{array}$$

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$$y = \begin{cases} 0 & \text{if the first constraint holds} \\ 1 & \text{if the second constraint holds} \end{cases}$$

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$$\sum_{j \in J} a_{1j} x_j \leq b_1 + M_1 y$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_2(1 - y)$$

$$y \in \{0, 1\}$$

If-then constraints

$$\begin{array}{l} \text{If } \sum_{j \in J} a_{1j} x_j \leq b_1 \\ \text{then } \sum_{j \in J} a_{2j} x_j \leq b_2 \end{array}$$

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$$\sum_{j \in J} a_{1j}x_j > b_1 \quad \text{or} \quad \sum_{j \in J} a_{2j}x_j \leq b_2$$

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$$\sum_{j \in J} a_{1j} x_j \geq b_1 + \varepsilon \quad \text{or} \quad \sum_{j \in J} a_{2j} x_j \leq b_2$$

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$$\begin{array}{l} \sum_{j \in J} a_{1j}x_j \leq b_1 - M_1y \\ \sum_{j \in J} a_{2j}x_j \leq b_2 + M_2(1 - y) \\ y \in \{0, 1\} \end{array}$$

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Product x_1x_2 , with $x_1, x_2 \in \{0, 1\}$.

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$$y \geq x_2 - u(1 - x_1)$$

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Special ordered sets (SOS1 and SOS2)

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Many MIP solvers are good at dealing with SOS1 variables.

SOS2 are positive variables that as a set can only have **two** non-zeros.

Piecewise linear objective

Non-linear objective:

$$\begin{array}{ll} \min & \sum_{j \in J} f_j(x_j) \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \begin{array}{l} \leq \\ \geq \end{array} b_i & \forall i \\ & x_j \geq 0 & \forall j \end{array}$$

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Trick when $f(x)$ is separable, i.e. $f(x) = \sum_{j \in J} f_j(x_j)$.

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Example:

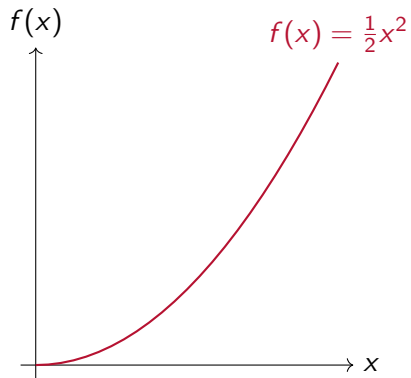
$$f(x) = x_1^2 + x_2 + e^{x_3}$$

Not separable:

$$f(x) = x_1 x_2 x_3$$

Piecewise linear objective

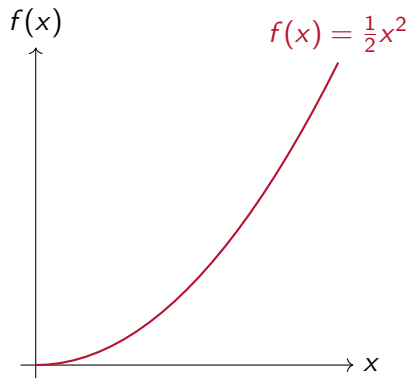
$$f(x) = \frac{1}{2}x^2$$



Piecewise linear objective

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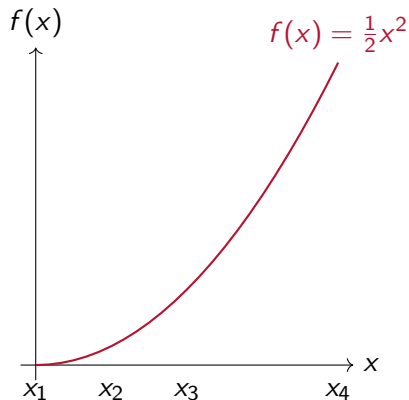
- ▶ Construct an approximation



Piecewise linear objective

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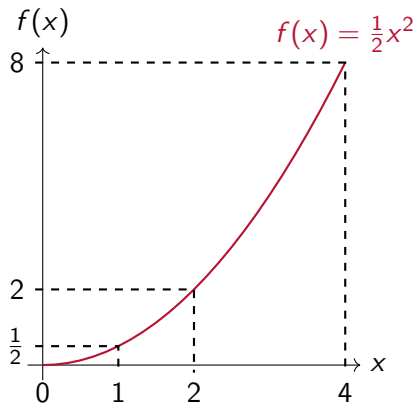
- ▶ Construct an approximation
- ▶ Choose breakpoints
 x_1, x_2, x_3, x_4



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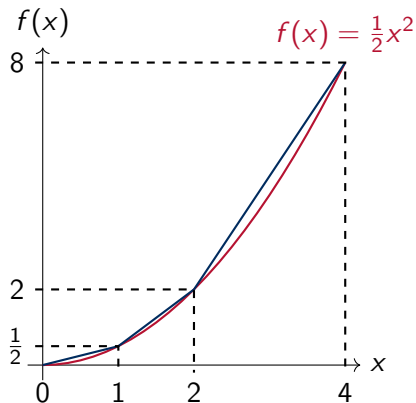
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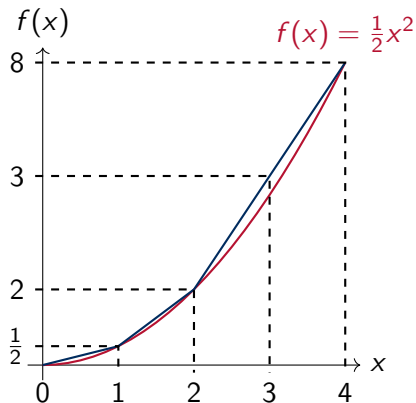
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- ▶ Approximate the interval between x_i and x_{i+1} by a linear function



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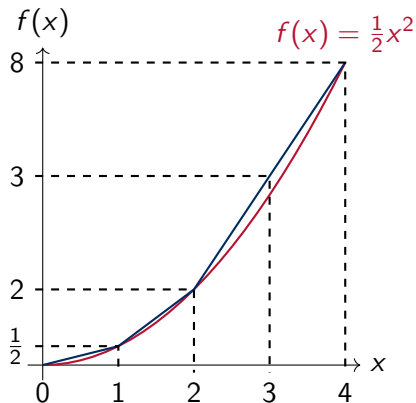
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- ▶ Now
 $\tilde{f}(x) = \lambda_i f(x_i) + \lambda_{i+1} f(x_{i+1})$
for $x = \lambda x_i + (1 - \lambda)x_{i+1}$.



Piecewise linear objective

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- ▶ Construct an approximation
- ▶ Choose breakpoints x_1, x_2, x_3, x_4
- ▶ Approximate the interval between x_i and x_{i+1} by a linear function
- ▶ Now
$$\tilde{f}(x) = \lambda_i f(x_i) + \lambda_{i+1} f(x_{i+1})$$
 for $x = \lambda x_i + (1 - \lambda)x_{i+1}$.
- ▶ $3 = \frac{1}{2}2 + \frac{1}{2}4$, thus $\tilde{f}(3) = \frac{1}{2}f(2) + \frac{1}{2}f(4) = 1 + 4 = 5$



Piecewise linear objective

$$\tilde{f}(x) = \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3) + \lambda_4 f(x_4)$$

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4$$

$$1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

λ_i 's are SOS2 variables.