Hands-on Tutorial on Optimization

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Modeling tools

Integer linear programming

$$
\begin{aligned}
\min \quad & \sum_{j \in J} c_j x_j & & \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \;\substack{\leq \\ = \\ \geq}\; b_i & & \forall i \\
& x_j \geq 0 & & \forall j \\
& x_j \in \mathbb{Z} & & \forall j
\end{aligned}
$$

Binary variables: Selection

Use cases

- Select one item out of the set J

$$x_j = \begin{cases} 1 & \text{if } j \text{ selected} \\ 0 & \text{otherwise.} \end{cases}$$

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$$\sum_{j \in J} x_j = 1$$

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- Construct a subset $S \subseteq J$ with certain properties

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$B \subseteq J$ blue items (at least two)
$R \subseteq J$ round items (at most three)

$$\sum_{j \in B} x_j \geq 2$$

$$\sum_{j \in R} x_j \leq 3$$

Binary variables: Assignment

Example

Given:	J set of n jobs
	I set of m machines
Task:	Assign each job to exactly one machine

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Task: Assign each job to exactly one machine

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Binary variables: Assignment

Example

Given: J set of n jobs
I set of m machines

Task: Assign each job to exactly one machine

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ assigned to } i \\ 0 & \text{otherwise.} \end{cases}$$

$$\sum_{i \in I} x_{ij} = 1$$

Binary variables: Assignment

Example

Given: Alice, Bob, Charly, Donald willing to pay c_{ij} for one of the 3 single-guest hotel rooms $i = 1, \ldots, 3$

Task: Maximize the revenue of the hotel owner

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$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ gets to stay in room } i, \\ 0 & \textit{otherwise}. \end{cases}$$

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$$\begin{aligned} \max \quad & \sum_{j \in J, i \in I} c_{ij} x_{ij} & \\ \text{s.t.} \quad & \sum_{i \in I} x_{ij} \leq 1 & \forall j \in J \\ & \sum_{j \in J} x_{ij} \leq 1 & \forall i \in I \\ & x_{ij} \in \{0, 1\} & \forall i, j \end{aligned}$$

Binary variables: Not

$$x_j = \begin{cases} 1 & \text{if something happens to } j, \\ 0 & \text{otherwise} = \text{nothing happens to } j. \end{cases}$$

$$1 - x_j = \begin{cases} 1 & \text{if nothing happens to } j \\ 0 & \text{if something happens to } j. \end{cases}$$

Binary variables: Assignment

Example

Given:	Alice, Bob, Charly, Donald
	willing to pay $c_{*,i}$ for one of the
	3 hotel rooms $i = 1, \ldots, 3$
	Alice and Charly do only stay if the other does not
Task:	Maximize the revenue of the hotel owner

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Reminder:

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ gets to stay in room } i, \\ 0 & \textit{otherwise}. \end{cases}$$

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$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ gets to stay in room } i, \\ 0 & otherwise. \end{cases}$$

Add the following inequality:

$$\sum_{i \in I} x_{iA} \leq 1 - \sum_{i \in I} x_{iC}$$

Binary variables: Conditional Assignment

Use cases

- Only assign j to i if i is open/exists/...

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ is assigned to } i \\ 0 & \text{otherwise.} \end{cases} \qquad y_j = \begin{cases} 1 & \text{if } i \text{ is open/exists/...} \\ 0 & \text{otherwise.} \end{cases}$$

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Coupling:

$$x_{ij} \leq y_i$$

Binary variables: Conditional Assignment

Use cases

- Only assign j to i if i is open/exists/...
- Ship z_{ij} from i to j if i is open/exists/...

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Coupling:

$$x_{ij} \leq y_i$$
$$z_{ij} \leq k_i y_i$$

Binary variables: Conditional Assignment

Use cases

- Only assign j to i if i is open/exists/...
- Ship z_{ij} from i to j if i is open/exists/...
- Observe capacity constraints

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ is assigned to } i \\ 0 & \text{otherwise.} \end{cases} \qquad y_j = \begin{cases} 1 & \text{if } i \text{ is open/exists/...} \\ 0 & \text{otherwise.} \end{cases}$$

Coupling:

$$x_{ij} \leq y_i$$
$$z_{ij} \leq k_i y_i$$
$$\sum_{j \in J} d_j x_{ij} \leq k_i y_i$$

Binary variables: Time indexing

Use cases

- Production planning (fixed processing times p_j)

$$x_{jt} = \begin{cases} 1 & \text{if } j \text{ is processed in } [t-1, t) \\ 0 & \textit{otherwise}. \end{cases}$$

Process at most k jobs simultaneously

Binary variables: Time indexing

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Process at most k jobs simultaneously

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Schedule consecutively

$$x_{j,t} \geq x_{j,t-1} - x_{j,t-p_j}$$

Binary variables: Time indexing

Use cases

- Scheduling maintenance for power plants

$$x_{jt} = \begin{cases} 1 & \text{if } j \text{ is active in } [t-1, t) \\ 0 & \textit{otherwise}. \end{cases}$$

$$y_{jt} = \begin{cases} 1 & \text{if } j \text{ is activated in } [t-1, t) \\ 0 & \textit{otherwise}. \end{cases}$$

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$$x_{j,t} \leq x_{j,t-1} + y_{j,t-1}$$
$$x_{j,t} \leq 1 - z_{j,t}$$

Tools

- Use the properties of binary variables
- Combine linear with binary variables
- Get creative!