Hands-on Tutorial on Optimization

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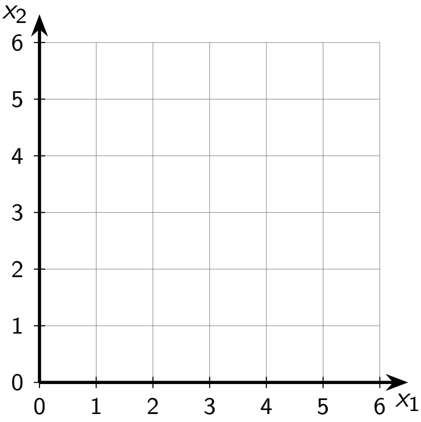
Geometry of Linear Programs

The Real, n-dimensional Space

$\mathbb{R}^n := \{x = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$

Elements $x \in \mathbb{R}^n$ can be seen as



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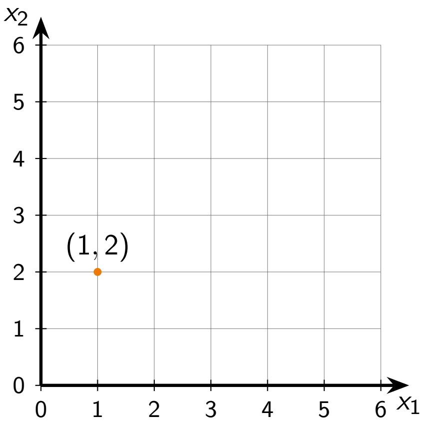

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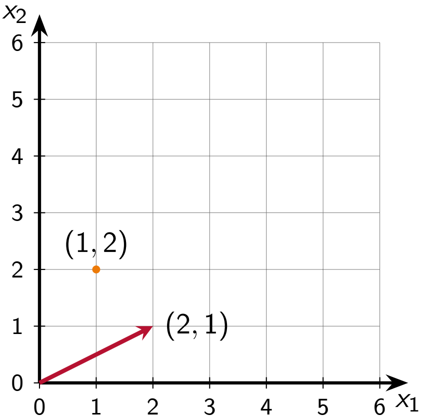
- Vectors



The Real, *n*-dimensional Space

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Elements $x \in \mathbb{R}^n$ can be seen as

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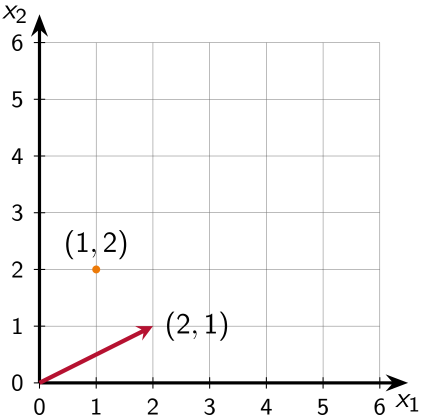
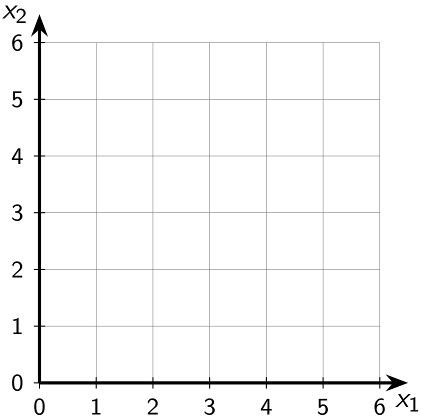
A *n*-tuple may represent the net profit of *n* different goods, or their inventory level, or the cost of production, etc.



Linear Equations

Let $a \in \mathbb{R}^n$ be a profit vector.

If you want the profit to be exactly b, how much should you produce?



Linear Equations

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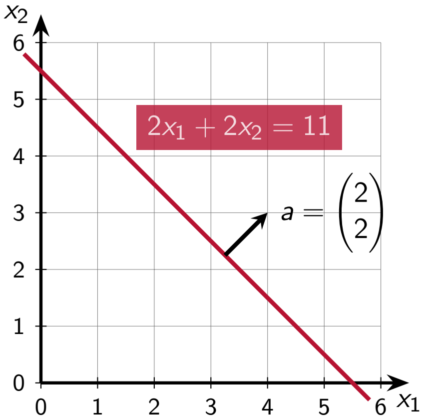
The answer is given as the set $\{x \in \mathbb{R}^n : a_1x_1 + \ldots + a_nx_n = b\}$.

This is a line in $\mathbb{R}^n$.



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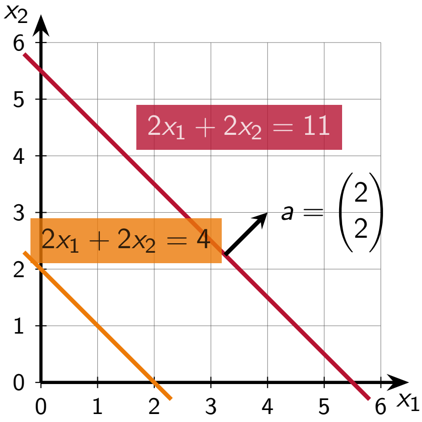
Changing the right hand side b corresponds to "moving" the line along the vector a.



Linear Inequalities

Let $a \in \mathbb{R}^n$ be a profit vector.

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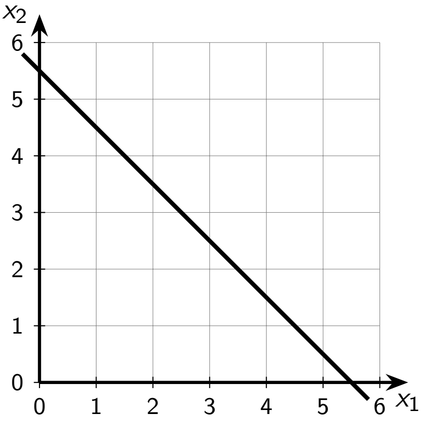

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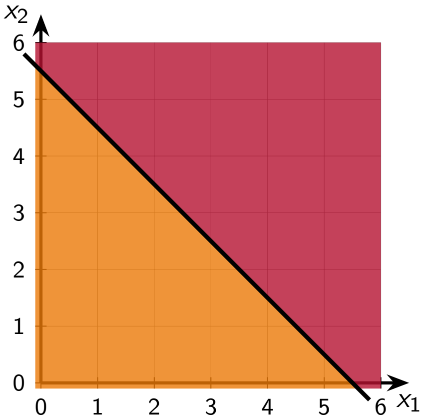
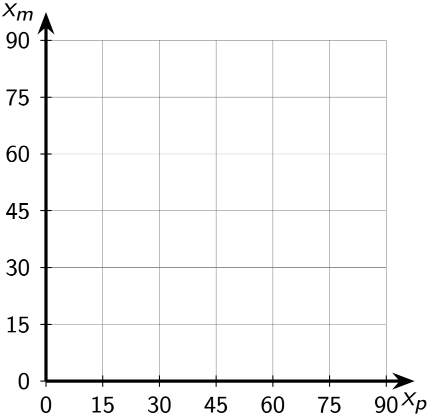
This is a halfspace in $\mathbb{R}^n$.



Graphical Solution of LPs

Consider the chips factory problem.

$$\begin{array}{rrcrcl} \max & 2x_p & + & \frac{3}{2}x_m & & \\ \text{s.t.} & 2x_p & + & 4x_m & \leq & 345 \\ & 4x_p & + & 5x_m & \leq & 480 \\ & 4x_p & + & 2x_m & \leq & 330 \\ & & & x_p,\ x_m & \geq & 0 \end{array}$$



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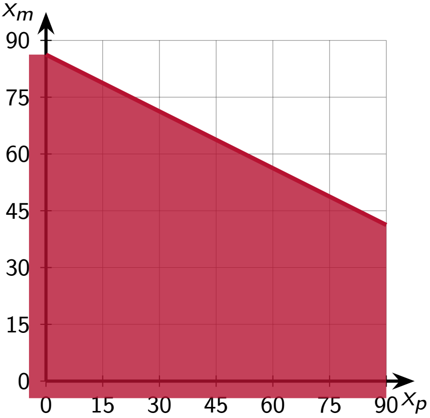

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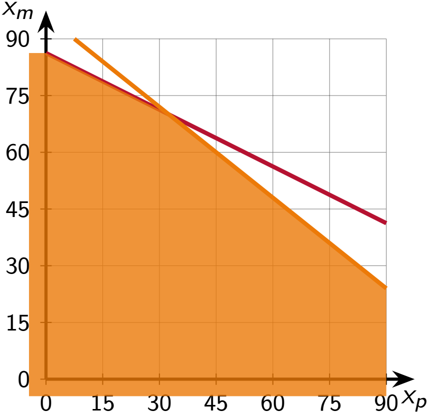

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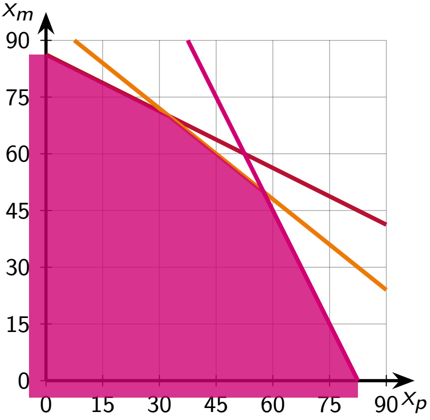

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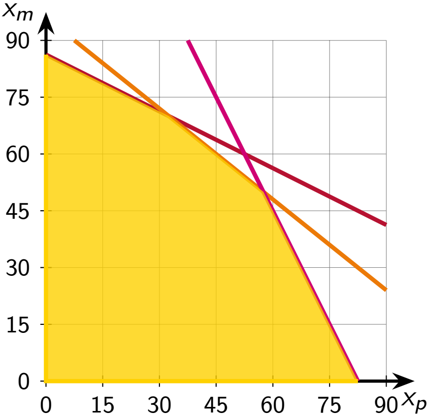

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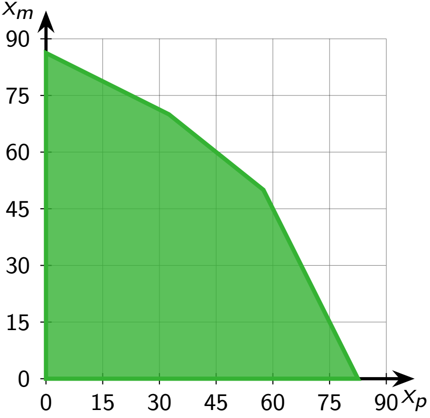

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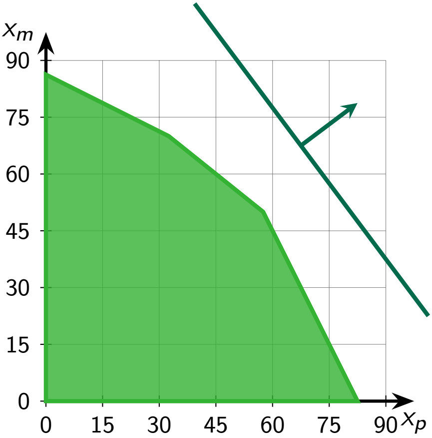

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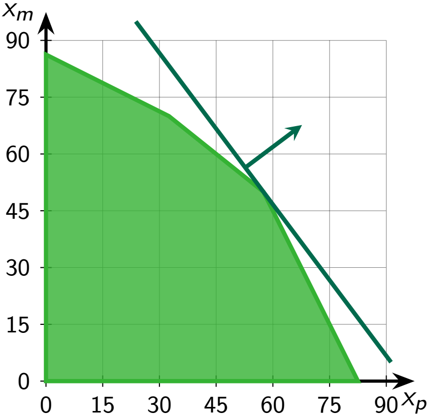

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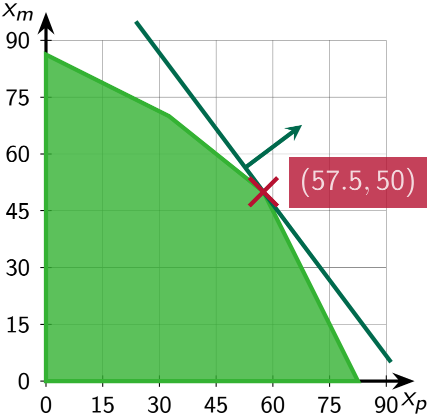

Optimal solution:
$(x_p, x_m) = (57.5, 50)$, s.t.
$2 \cdot 57.5 + \frac{3}{2} \cdot 50 = 190$.

Graphical Solution of LPs

Consider the crude oil processing problem.

Processes:

Process	cost per barrel	output per 10 barrels
1	3€	2 barrels heavy oil
		2 barrels medium heavy oil
		1 barrel light oil
2	5€	1 barrel heavy oil
		2 barrels medium heavy oil
		4 barrels light oil

Demands:	heavy oil:	3 barrels
	medium heavy oil:	5 barrels
	light oil:	4 barrels

Construct the polyhedral representation and solve the problem then graphically by hand.

Graphical Solution of LPs

Consider the crude oil processing problem.

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\begin{aligned}
\min \quad & 30x_1 + 50x_2 \\
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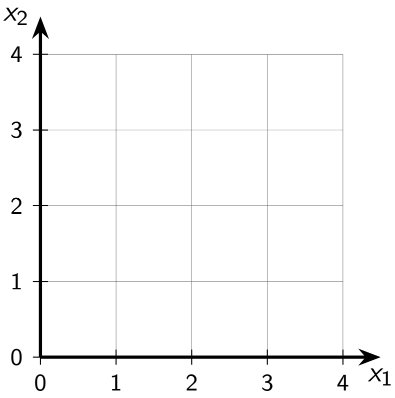

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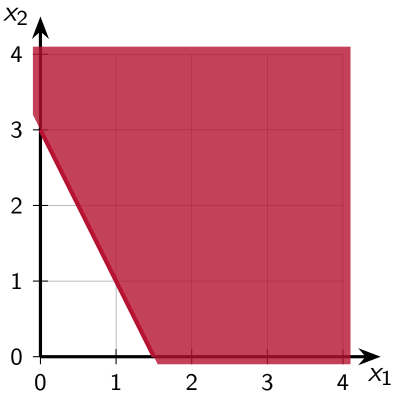

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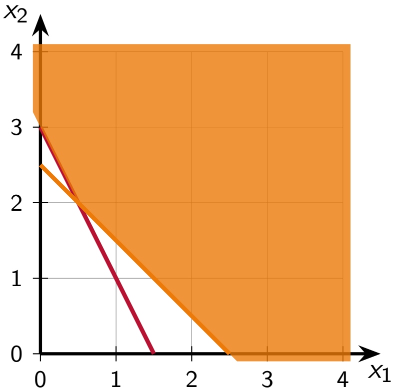

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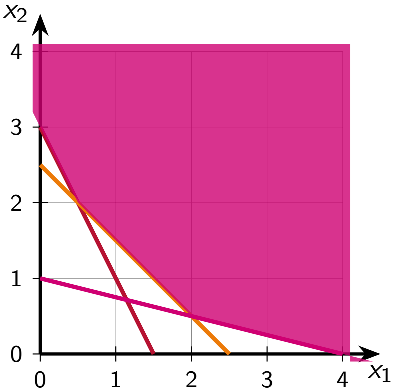

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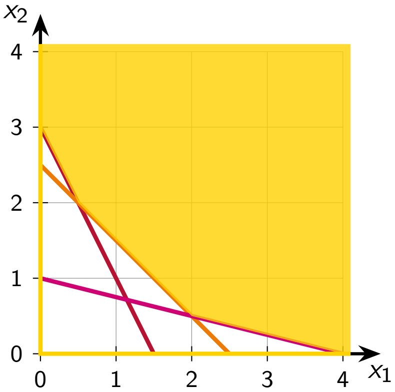

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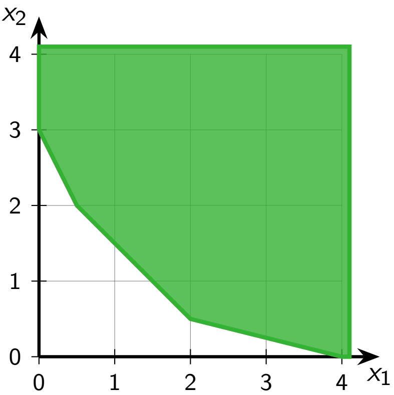

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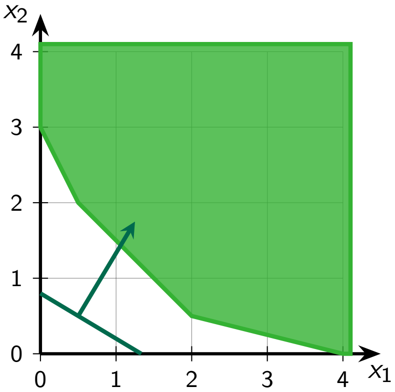

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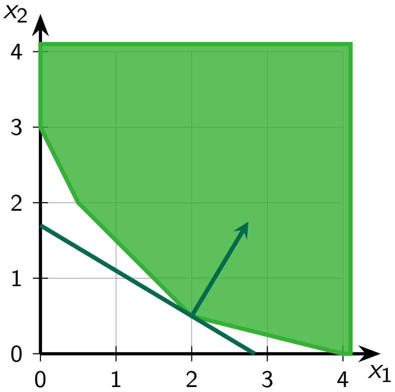

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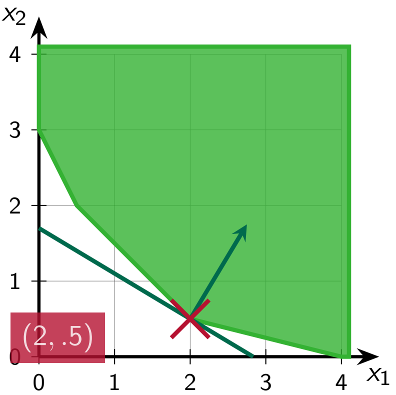

Optimal solution: $(x_1, x_2) = (2, .5)$,
s.t. $30 \cdot 2 + 50 \cdot .5 = 85$.

Extreme points

Definition

Let $M \neq \emptyset$ convex. A point $x \in M$ is an extreme point of M if we cannot find two points $y, z \in M \setminus \{x\}$ and a scalar $\lambda \in (0, 1)$ such that

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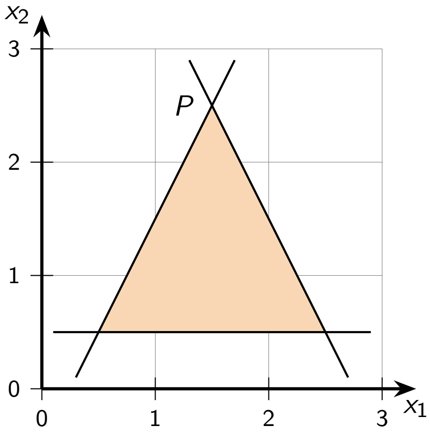
- Examples of convex sets with infinitely many extreme points: circle or ball

Example

$$P = \left\{ x \in \mathbb{R}^2 : \begin{array}{rl} x_2 \geq & \frac{1}{2} \\ 2x_1 + x_2 \leq & \frac{11}{2} \\ -2x_1 + x_2 \leq & -\frac{1}{2} \end{array} \right\}$$



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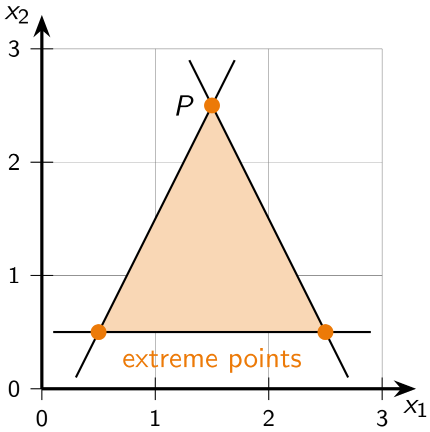

Matrices

A $m \times n$ dimensional matrix A is an array of $n \cdot m$ real numbers $a_{i,j}$:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

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The transposed of A is defined as follow:

$$A^T := \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

Special Matrices

Let $\mathbf{0} \in \mathbb{R}^{m \times n}$ denote the matrix that only contains 0.

By $\mathbf{1} \in \mathbb{R}^{n \times n}$ we denote the matrix that only consists of 0s except the diagonal:

$$\mathbf{1} := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Multiplication

Let $x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Then, $c^T x := \sum_{j=1}^n c_j \cdot x_j$, and

$$Ax = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \sum_{j=1}^n a_{1,j} \cdot x_j \\ \sum_{j=1}^n a_{2,j} \cdot x_j \\ \vdots \\ \sum_{j=1}^n a_{m,j} \cdot x_j \end{pmatrix} \in \ R^m$$

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Thus,

$$\max c^T x \qquad Ax \leq b \qquad \Leftrightarrow \qquad \max \sum_{j=1}^{n} c_j \cdot x_j$$

$$\begin{matrix} \sum_{j=1}^n a_{1,j} \cdot x_j & \leq & b_1 \\ \sum_{j=1}^n a_{2,j} \cdot x_j & \leq & b_2 \\ \vdots & \vdots & \vdots \\ \sum_{j=1}^n a_{m,j} \cdot x_j & \leq & b_3 \end{matrix}$$

Example

$$P = \left\{ x \in \mathbb{R}^2 : \begin{array}{rcrcr} & & x_2 & \geq & \frac{1}{2} \\ 2x_1 & + & x_2 & \leq & \frac{11}{2} \\ -2x_1 & + & x_2 & \leq & -\frac{1}{2} \end{array} \right\}$$

$$= \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 0 & -1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} x \leq \begin{pmatrix} -\frac{1}{2} \\ \frac{11}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}$$

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Example

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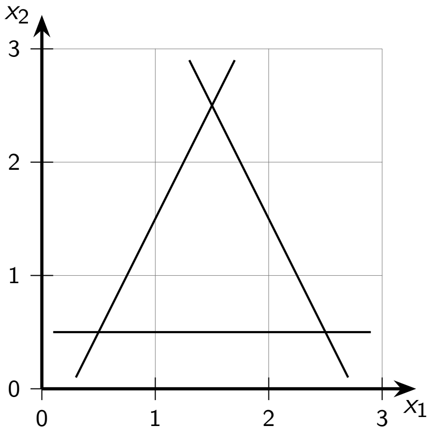

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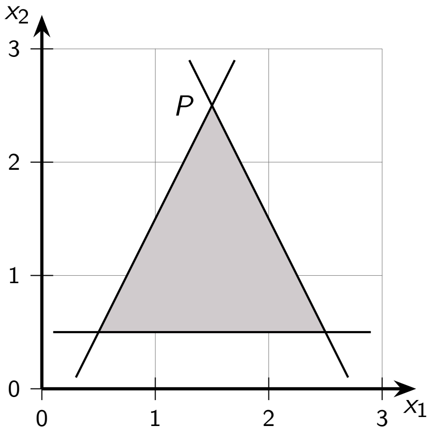

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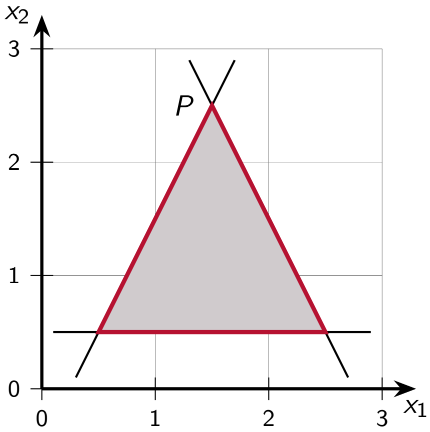

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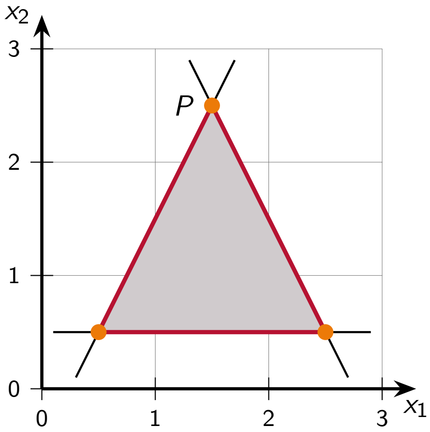
Vertices of P



Optimal Solutions

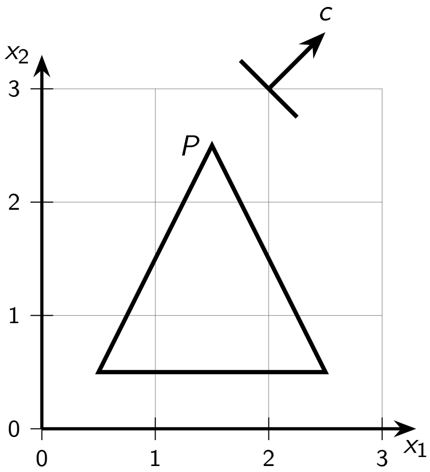
Let $P = P(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ a polyhedron.

Theorem

The optimum of $\min\{c^T x \mid x \in P\}$ is attained at a vertex of P.

Example

$$\begin{array}{lrcrcr} \min & -x_1 & - & x_2 & & \\ \text{s.t.} & & & x_2 & \geq & \frac{1}{2} \\ & 2x_1 & + & x_2 & \leq & \frac{11}{2} \\ & -2x_1 & + & x_2 & \leq & -\frac{1}{2} \end{array}$$



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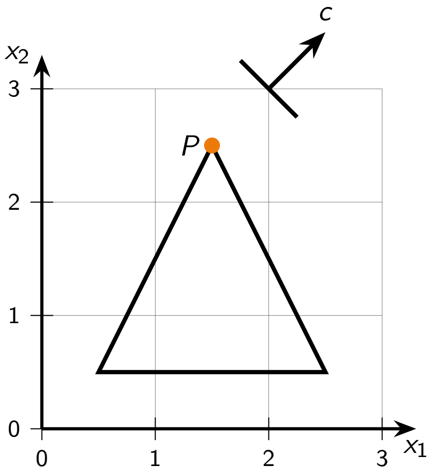

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Example: Unit cube $P = \{x \in \mathbb{R}^n \,|\, 0 \leq x_i \leq 1, i = 1, \ldots, n\}$

Number of inequalities: $m = 2n$
Number of vertices: 2^n

Works well in practice.


Break