Hands-on Tutorial on Optimization

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Duality

Motivation

$$\begin{aligned} \min\ & 7x_1 + 3x_2 && =: \ z(x) \\ \text{s.t.}\ & x_1 + x_2 && \geq 2 && (1) \\ & 3x_1 + x_2 && \geq 4 && (2) \\ & x_1, x_2 && \geq 0 && (3) \end{aligned}$$

A feasible solution:
$x_1 = x_2 = 1$ with $z(x) = 10$.
How close to optimum?

Goal: Find lower bound on the optimum.

Ineq. (1),(3) imply: $z(x) = 7x_1 + 3x_2 \geq x_1 + x_2 \geq 2 \Rightarrow \mathrm{OPT} \geq 2$

Ineq. (2),(3) imply: $z(x) = 7x_1 + 3x_2 \geq 3x_1 + x_2 \geq 4 \Rightarrow \mathrm{OPT} \geq 4$

Idea: linear combination of constraints with coefficients $y_1 = 1$ and $y_2 = 2$, that is, $z(x) \geq y_1 \cdot (1) + y_2 \cdot (2)$.

$$z(x) = 7x_1 + 3x_2 \geq 1 \cdot (x_1 + x_2) + 2 \cdot (3x_1 + x_2) \geq 1 \cdot 2 + 2 \cdot 4 = 10.$$

Hence, the above solution is optimal. ... now generalize.

Motivation

$$\begin{aligned} \min\ & 7x_1 + 3x_2 & =: z(x) & \\ \text{s.t.}\ & x_1 + x_2 & \geq 2 & \quad (1) \\ & 3x_1 + x_2 & \geq 4 & \quad (2) \\ & x_1, x_2 & \geq 0 & \quad (3) \end{aligned}$$

Find $y_1 \geq 0$ and $y_2 \geq 0$ with

$$\begin{aligned} z(x) & \geq y_1(x_1 + x_2) + y_2(3x_1 + x_2) \\ & \geq y_1 \cdot 2 + y_2 \cdot 4 \end{aligned}$$

maximizing the right hand side.
It must hold: $y_1 + 3y_2 \leq 7$ and $y_1 + y_2 \leq 3$. It is again an LP.

Primal LP

$$\begin{aligned} \min\ & 7x_1 + 3x_2 & \\ \text{s.t.}\ & x_1 + x_2 & \geq 2 \\ & 3x_1 + x_2 & \geq 4 \\ & x_1, x_2 & \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max\ & 2y_1 + 4y_2 & \\ \text{s.t.}\ & y_1 + 3y_2 & \leq 7 \\ & y_1 + y_2 & \leq 3 \\ & y_1, y_2 & \geq 0 \end{aligned}$$

Primal and Dual Program

Arbitrary linear program:

$$
\begin{array}{lll}
\min & c^T x & \\
\text{s.t.} & a_i^T x \geq b_i & \text{for } i \in M_1 \\
 & a_i^T x \leq b_i & \text{for } i \in M_2 \\
 & a_i^T x = b_i & \text{for } i \in M_3 \\
 & x_j \geq 0 & \text{for } j \in N_1 \\
 & x_j \leq 0 & \text{for } j \in N_2 \\
 & x_j \text{ free} & \text{for } j \in N_3
\end{array}
$$

Obtain lower bound:

$$
\begin{array}{lll}
\max & y^T b & \\
\text{s.t.} & y_i \geq 0 & \text{for } i \in M_1 \\
 & y_i \leq 0 & \text{for } i \in M_2 \\
 & y_i \text{ free} & \text{for } i \in M_3 \\
 & y^T A_j \leq c_j & \text{for } j \in N_1 \\
 & y^T A_j \geq c_j & \text{for } j \in N_2 \\
 & y^T A_j = c_j & \text{for } j \in N_3
\end{array}
$$

The linear program on the right is the dual linear program of the primal linear program on the left.

Examples

primal LP (min)	dual LP (max)
$\min\ c^T x$ s.t. $Ax \geq b$ $x \geq 0$	$\max\ b^T y$ s.t. $A^T y \leq c$ $y \geq 0$
$\min\ c^T x$ s.t. $Ax = b$ $x \geq 0$	$\max\ b^T y$ s.t. $A^T y \leq c$

Theorem. The dual LP of a dual LP is the primal LP.

Primal & Dual Variables & Constraints

	primal LP (min)	dual LP (max)	
Constraints	$\geq b_i$	≥ 0	Variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
Variables	≥ 0	$\leq c_i$	Constraints
	≤ 0	$\geq c_i$	
	free	$= c_i$	

$\rightarrow$ Example

Duality Theorems

Weak Duality

primal (P) $\min\ c^T x$ s.t. $Ax \geq b$, $x \geq 0$

dual (D) $\max\ b^T y$ s.t. $A^T y \leq c$, $y \geq 0$

Theorem

Let $\bar{x}$ be feasible a feasible solution for the primal LP (P) and let $\bar{y}$ be a feasible solution for the dual LP (D). Then

$$c^T \cdot \bar{x} \ \geq \ \bar{y}^T \cdot b.$$

Proof. ...

Implications ...

1. If (P) is unbounded ($\text{Opt} = -\infty$), then (D) is infeasible.
2. If (D) is unbounded ($\text{Opt} = \infty$), then (P) is infeasible.
3. Let $\bar{x}$ and $\bar{p}$ be feasible solutions for (P) and (D) with $c^T \cdot \bar{x} = \bar{p}^T \cdot b$, then $\bar{x}$ and $\bar{p}$ are optimal.

Strong Duality

Theorem

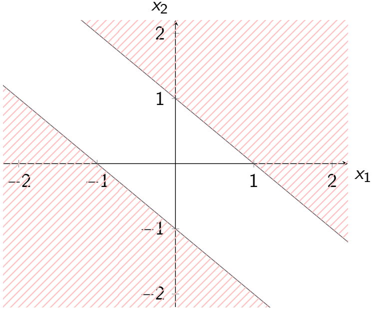
If the primal LP has an optimal solution x^*, then there exists an optimal solution y^* for the dual LP and $c^T x^* = b^T y^*$.

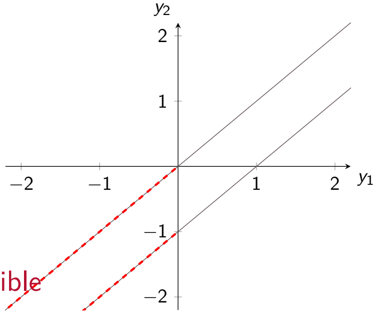
Possible primal-dual pairs:

primal\dual	optimal	unbounded	infeasible
optimal	strong duality	impossible	impossible
unbounded	impossible	impossible	(1) weak duality
infeasible	impossible	(1) weak duality	(2) possible, c.f. Ex.

Example I

$$
\begin{array}{lrcrcr l}
\max & x_1 & & & & & \\
\text{s. t.} & x_1 & + & x_2 & \geq & 1 & \mid y_1 \\
 & -x_1 & - & x_2 & \geq & 1 & \mid y_2 \\
 & & & x_1,\, x_2 & \in & \mathbb{R} &
\end{array}
\qquad
\begin{array}{lrcrcr l}
\min & y_1 & + & y_2 & & & \\
\text{s. t.} & y_1 & - & y_2 & = & 1 & \mid x_1 \\
 & y_1 & - & y_2 & = & 0 & \mid x_2 \\
 & & & y_1,\, y_2 & \leq & 0 &
\end{array}
$$



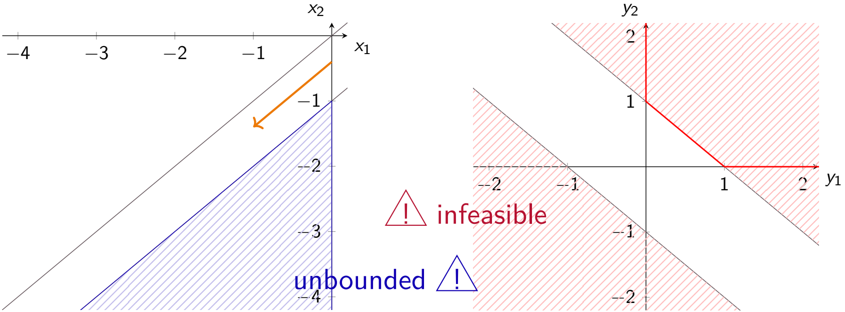


⚠ both infeasible

Example II

$$\begin{array}{lrcrcrl} \min & x_1 & + & x_2 & & & \\ \text{s. t.} & x_1 & - & x_2 & \geq & 1 & \mid y_1 \\ & x_1 & - & x_2 & \geq & 0 & \mid y_2 \\ & & & x_1, x_2 & \leq & 0 & \end{array}$$

$$\begin{array}{lrcrcrl} \max & y_1 & & & & & \\ \text{s. t.} & y_1 & + & y_2 & \geq & 1 & \mid x_1 \\ & -y_1 & - & y_2 & \geq & 1 & \mid x_2 \\ & & & y_1, y_2 & \geq & 0 & \end{array}$$



An application of duality

Interactive: Minimal Vertex Cover

Problem: Min Vertex Cover

Given: Graph $G = (V, E)$

Task: Find a minimal subset $V' \subseteq V$ such that each edge $e = \{u, v\} \in E$ has an endpoint in V'; we say e is covered.

Integer LP: Decision variable $x_v \in \{0, 1\}$ indicates if $v \in V'$.

$$\begin{aligned} \min \ & \sum_{v \in V} x_v \ =: z \\ \text{s.t.} \ & x_u + x_v \geq 1, \quad \text{for all } \{u, v\} \in E \\ & x_v \in \{0, 1\}, \ \text{for all } v \in V. \end{aligned}$$

LP relaxation: Replace $x_v \in \{0, 1\}$ by $x_v \geq 0$.

Observation: $z_{\text{LP}} \leq z_{\text{ILP}}$ (Any ILP solution is feasible for the LP.)

Dual LP for Minimal Vertex Cover

The dual LP for the LP relaxation:

$$\begin{aligned} \max \sum_{e \in E} y_e & \\ \text{s.t.} \sum_{e \in \delta(v)} y_e \leq 1, & \qquad \text{for all } v \in V \\ y_e \geq 0, & \qquad \text{for all } e \in E. \end{aligned}$$

For $v \in V$ let $\delta(v) := \{e \in E \mid e = (u, v), u \in V\}$ the set of edges that are incident with v.

Observation. LP Relaxation of the Maximal Matching.

Problem: Max Matching	
Given:	Graph $G = (V, E)$
Task:	Find a maximal matching, i.e., a maximal subset $M \subseteq E$ such that any vertex is incident to at most one edge $e \in M$.

Interactive: König's Theorem

Theorem

There are at least as many vertices in a minimal vertex cover as there are edges in a maximal matching.

Proof. Follows by weak duality: $z_{\text{VC}} \geq z_{\text{VC}}^{\text{LP}} = z_{\text{M}}^{\text{LP}} \geq z_{\text{M}}$.

Definition. A Graph $G = (V, E)$ is bipartite, if there exists a partition $V = L \cup R$ such that there are no edges between L and R, i.e., there exists no $\{u, v\} \in E$ with $u \in L$ and $v \in R$.

Theorem (König, 1931)

In a bipartite graph it holds that the number of vertices in a minimal vertex cover **equals** the number of edges in a maximal matching.

Important: In general, weak and strong duality only hold for LPs (relaxations). However, it can be shown that the vertex cover and matching LPs always have an integral solution if the graph is bipartite.

Complementary slackness

Complementary Slackness

Consider an arbitrary primal-dual pair (P) and (D):

$$(P)\ \min\ c^T \cdot x \quad \text{s.t.}\ A \cdot x \geq b,\ x \geq 0 \qquad\qquad (D)\ \max\ y^T \cdot b \quad \text{s.t.}\ A^T \cdot y \leq c,\ y \geq 0$$

Theorem

Let $\bar{x}$ be feasible for (P) and $\bar{y}$ feasible for (D). Then, $\bar{x}$ and $\bar{y}$ are optimal if and only if

$$\bar{x}_i \cdot (c_i - (A^T \cdot \bar{y})_i) = 0, \qquad \text{for all } i, \text{ and}$$

$$\bar{y}_j \cdot (b_j - (A \cdot \bar{x})_j) = 0, \qquad \text{for all } j.$$

The theorem holds for arbitrary prima-dual pairs $(P), (D)$.

Observation. In optimal solution: either, a variable vanishes $(= 0)$, or, the corresponding dual inequality is tight $(=)$. A free variable corresponds to an equation in the dual that is tight by definition.