Hands-on Tutorial on Optimization

F. Eberle, R. Hoeksma, and N. Megow

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Column Generation

Cutting Stock

Problem: Cutting Stock

Given: Length $W \in \mathbb{N}$
n orders with
demand $d_i \in \mathbb{N}$ and a
length $w_i \in \mathbb{N}$ with $w_i \leq W$

Task: Minimize the number of rolls of length W
while satisfying the demand

Interactive: A First LP for Cutting Stock

Definition (Cutting Pattern)

A cutting pattern is a vector $x \in \mathbb{N}_0^n$, where x_i denotes the number of short rolls of length w_i cut out of a long roll.

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Binary variables $y^{(k)}$ indicating whether roll k is needed

Cutting pattern $x^{(k)}$ for roll k

Formulate an ILP for the cutting stock problem!

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$$
\begin{aligned}
\min \quad & \sum_{k=1}^{K} y^{(k)} & & \\
\text{s.t.} \quad & \sum_{k=1}^{K} x_i^{(k)} \geq d_i & \text{for all } i = 1, \dots, n & \quad \text{(demand)} \\
& \sum_{k=1}^{K} w_i x_i^{(k)} \leq y^{(k)} W & \text{for all } k = 1, \dots, K & \quad \text{(valid pattern)} \\
& y^{(k)} \in \{0, 1\} & \text{for all } k = 1, \dots, K & \\
& x^{(k)} \in \mathbb{Z}^n & \text{for all } k = 1, \dots, K &
\end{aligned}
$$

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Observation: Problems in cutting plane and branch & bound approaches.

$\Rightarrow$ Try a different ILP formulation.

Interactive: A Second LP for Cutting Stock

$\mathcal{Q} := \{x \in \mathbb{N}_0^n : \sum_{i=1}^n w_i x_i \leq W\}$

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Enumerate $\mathcal{Q} = \{x^{(1)}, x^{(2)}, \ldots, x^{(|\mathcal{Q}|)}\}$.

Variable $u^{(j)}$ denotes how often cutting pattern $x^{(j)} \in \mathcal{Q}$ is used.

Formulate an ILP for the cutting stock problem!

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$$\min \quad \sum_{j=1}^{|\mathcal{Q}|} u^{(j)}$$

$$\text{s.t.} \quad \sum_{j=1}^{|\mathcal{Q}|} x_i^{(j)} u^{(j)} \geq d_i \quad \text{for all } i = 1, \ldots, n \qquad \text{(demand)}$$

$$u^{(j)} \in \mathbb{N}_0 \text{ for all } j = 1, \ldots, |\mathcal{Q}|$$

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⇒ Recall polyhedral description of LPs!

Idea of Column Generation

Observations:

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Duality theory

The Cutting Stock LP and its Dual

$$\min \quad \sum_{j=1}^{|\mathcal{Q}|} u^{(j)}$$

$$\text{s.t.} \quad \sum_{j=1}^{|\mathcal{Q}|} x_i^{(j)} u^{(j)} \geq d_i \ \forall \ i \in [n]$$

$$u^{(j)} \geq 0 \ \ \forall \ j \in [|\mathcal{Q}|]$$

The Cutting Stock LP and its Dual

$$\min \sum_{j=1}^{|\mathcal{Q}|} u^{(j)} \quad \Big| \quad \max$$

$$\text{s.t.} \sum_{j=1}^{|\mathcal{Q}|} x_i^{(j)} u^{(j)} \geq d_i \ \forall\ i \in [n] \quad \Big| \quad z_i$$

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$\min \quad \sum_{j=1}^{\lvert\mathcal{Q}\rvert} u^{(j)}$	max	$\max \quad \sum_{i=1}^{n} d_i z_i$
s.t. $\sum_{j=1}^{\lvert\mathcal{Q}\rvert} x_i^{(j)} u^{(j)} \geq d_i \ \forall\ i \in [n]$	z_i	
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Primal		Dual
$\min \quad \sum_{j=1}^{\lvert\mathcal{Q}\rvert} u^{(j)}$	max	$\max \quad \sum_{i=1}^{n} d_i z_i$
$\text{s.t.} \quad \sum_{j=1}^{\lvert\mathcal{Q}\rvert} x_i^{(j)} u^{(j)} \geq d_i \ \forall \ i \in [n]$	z_i	$\text{s.t.} \quad \sum_{i=1}^{n} x_i^{(j)} z_i \leq 1 \ \forall \ j \in [\lvert\mathcal{Q}\rvert]$
$u^{(j)} \geq 0 \ \ \forall \ j \in [\lvert\mathcal{Q}\rvert]$	$\leq$	

The Cutting Stock LP and its Dual

$\min \quad \sum_{j=1}^{\lvert\mathcal{Q}'\rvert} u^{(j)}$	max	$\max \quad \sum_{i=1}^{n} d_i z_i$
$\text{s.t.} \quad \sum_{j=1}^{\lvert\mathcal{Q}'\rvert} x_i^{(j)} u^{(j)} \geq d_i \ \forall\ i \in [n]$	z_i	$\text{s.t.} \quad \sum_{i=1}^{n} x_i^{(j)} z_i \leq 1 \ \forall\ j \in [\lvert\mathcal{Q}'\rvert]$
$u^{(j)} \geq 0 \ \ \forall\ j \in [\lvert\mathcal{Q}'\rvert]$	$\leq$	$z_i \geq 0 \ \forall\ i \in [n]$

Restrict to $\mathcal{Q}'$.

Recap: Duality Theory

- Basic primal solution corresponds to a basic dual solution
- They can be derived by complementary slackness
- $\mathcal{Q}' \subset \mathcal{Q}$ corresponds to a restriction of the dual constraints.
- A primal/dual pair is optimal iff
  1. their objective values agree (in this case always true) and
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Idea: To check if a solution over $\mathcal{Q}'$ is optimal over $\mathcal{Q}$ as well, we will calculate the corresponding dual solution for the restricted LP. Then, we check all dual constraints for feasibility.

If all constraints are feasible, we found an optimal solution.
Else, we add the violated constraint j for some cutting pattern $x^{(j)}$ to the dual which corresponds to adding the variable $u^{(j)}$ to the primal.

Finding a violated constraint

Let z^* be a dual solution.

Finding $j \in \mathcal{Q}$ such that

$$\sum_{i=1}^{n} x_i^{(j)} z_i^* - 1 > 0.$$

is equivalent to finding

$$m^* := \max_{j \in \mathcal{Q}} \sum_{i=1}^{n} x_i^{(j)} z_i^*$$

and checking whether $m^* > 1$.

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Observation: No need to completely enumerate $\mathcal{Q}$.
The Pricing problem should be efficiently solvable.

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Observation: No need to completely enumerate $\mathcal{Q}$.
The Pricing problem should be efficiently solvable.

If the optimal objective is greater than 1, we have found a cutting pattern that violates a dual inequality.

Column Generation: General Idea

1. Start with an LP and its dual

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \le b \quad \text{(P)} \\ & x \ge 0 \end{aligned} \qquad \begin{aligned} \min\ & b^T y \\ \text{s.t.}\ & A^T y \ge c \quad \text{(D)} \\ & y \ge 0 \end{aligned}$$

Restrict to $J' \subset [n]$ to obtain the reduced master problem

$$\begin{aligned} \max\ & (c')^T x' \\ \text{s.t.}\ & A'x' \le b \quad \text{(P')} \\ & x' \ge 0 \end{aligned} \qquad \begin{aligned} \min\ & b^T y \\ \text{s.t.}\ & (A')^T y \ge c' \quad \text{(D')} \\ & y \ge 0 \end{aligned}$$

2. Solve (P') and (D') to get an optimal primal/dual pair (x', y).

Column Generation: General Idea

3. Find a violated inequality of (D') by solving the pricing problem

$$\begin{aligned} \max_j \; & c_j - a^{(j)} y =: z \\ \text{s.t.} \quad & a^{(j)} \text{ is the } j\text{-th column of } A \end{aligned} \tag{Π}$$

4. If $z > 0$, add x_j to (P') and go to 2.

Otherwise, set $x'_j = 0$ if $j \notin J'$ and output (x', y).

An Example: Parameters

Parameters:

$$W = 70 \qquad w_1 = 17 \qquad d_1 = 6$$
$$w_2 = 15 \qquad d_2 = 2$$
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$$\mathcal{Q}' = \{(1,0,0), (0,1,0), (0,0,1)\}$$

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Reduced set $\mathcal{Q}' = \{(1,0,0),(0,1,0),(0,0,1)\}$

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Optimal solution $u = (6,2,1)$ with dual $z = (1,1,1)$.

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Pricing problem:

$$\begin{array}{ll} \max & x_1 + x_2 + x_3 - 1 \\ \text{s.t.} & 17x_1 + 15x_2 + 63x_3 \leq 70 \\ & x \in \mathbb{N}_0 \end{array}$$

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Possible solution $x = (4,0,0)$

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\end{array}
$$

Optimal solution $u = (0, 0, 1, \frac{6}{4}, \frac{1}{2})$ with dual $z = (\frac{1}{4}, \frac{1}{4}, 1)$.

Pricing problem:

$$
\begin{array}{ll}
\max & \frac{1}{4}x_1 + \frac{1}{4}x_2 + x_3 - 1 \\
\text{s.t.} & 17x_1 + 15x_2 + 63x_3 \leq 70 \\
& x \in \mathbb{N}_0
\end{array}
$$

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An Example: Step 3

Reduced set $\mathcal{Q}' = \{(1,0,0),(0,1,0),(0,0,1),(4,0,0),(0,4,0)\}$

$$\begin{array}{llr} \min & u_1 + u_2 + u_3 + u_4 + u_5 & \\ \text{s.t.} & u_1 + 4u_4 & \geq 6 \\ & u_2 + 4u_5 & \geq 2 \\ & u_3 & \geq 1 \\ & u & \geq 0 \end{array} \qquad \begin{array}{llr} \max & 6z_1 + 2z_2 + z_3 & \\ \text{s.t.} & z_1 & \leq 1 \\ & z_2 & \leq 1 \\ & z_3 & \leq 1 \\ & 4z_1 & \leq 1 \\ & 4z_2 & \leq 1 \\ & z & \geq 0 \end{array}$$

Optimal solution $u = (0,0,1,\frac{6}{4},\frac{1}{2})$ with dual $z = (\frac{1}{4},\frac{1}{4},1)$.

Pricing problem:

$$\begin{array}{llr} \max & \frac{1}{4}x_1 + \frac{1}{4}x_2 + x_3 - 1 & \\ \text{s.t.} & 17x_1 + 15x_2 + 63x_3 & \leq 70 \\ & x & \in \mathbb{N}_0 \end{array}$$

End