Hands-on Tutorial on Optimization

F. Eberle, R. Hoeksma, and N. Megow

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Branch & Bound

Branch & Bound: A General Framework for ILPs

- Introduced in the 1960's by Land and Doig
- Based on two principle ideas
  1. Branching
  2. Bounding
- Complete enumeration might be performed

A First Glance

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\quad & Ax \le b \\ & x \ge 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

Branching:

- Compute a solution x' of the current subproblem
- If $x' \in \mathbb{Z}^n$, compare to the current lower bound.
  - If better, store it as the new current best solution.
  - If worse, prune the current branch.
- If $x' \notin \mathbb{Z}^n$, choose $x'_i \notin \mathbb{Z}$ and create two subproblems
  - Add $x_i \le \lfloor x'_i \rfloor$.
  - Add $x_i \ge \lceil x'_i \rceil$.

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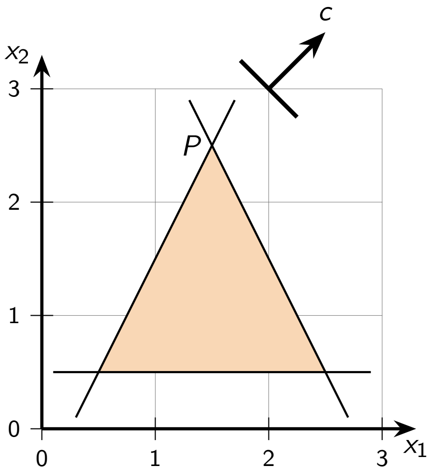
Bounding:

If $c^T x' \le L$, where x' is the LP solution of the current subproblem and L the current lower bound, the current branch can be pruned.

Example: Integer LP

$$\begin{array}{lrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & & & x_2 & \geq & \frac{1}{2} \\ & 2x_1 & + & x_2 & \leq & \frac{11}{2} \\ & -2x_1 & + & x_2 & \leq & -\frac{1}{2} \end{array}$$



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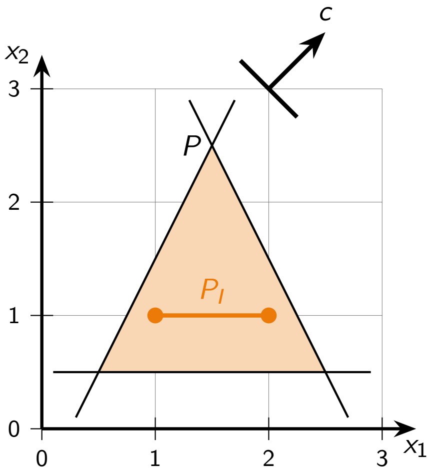
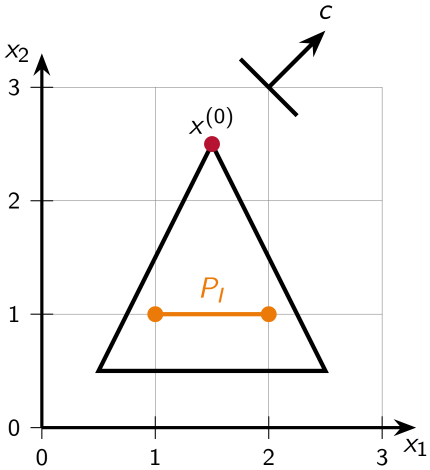


Example: Branching

1. solve LP relaxation



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1. solve LP relaxation
2. $x_i^{(0)} \notin \mathbb{Z} \rightarrow$ branching

$$x_i \leq \lfloor x_i^{(0)} \rfloor$$

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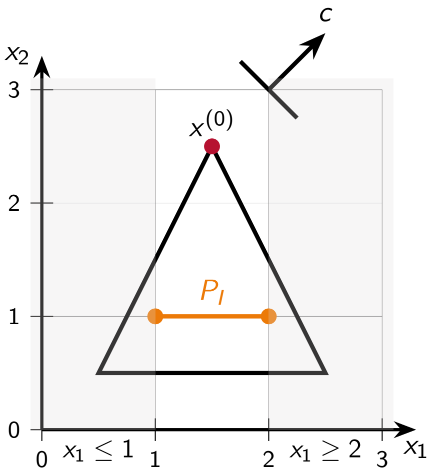

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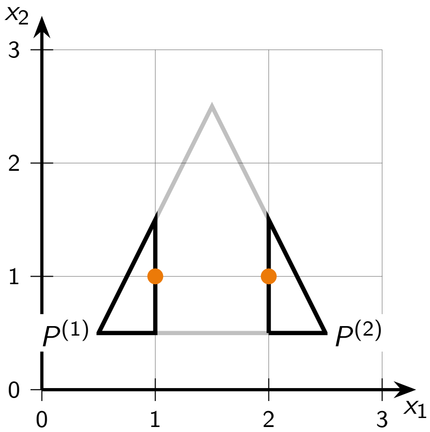
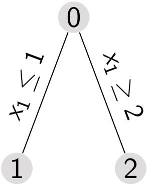
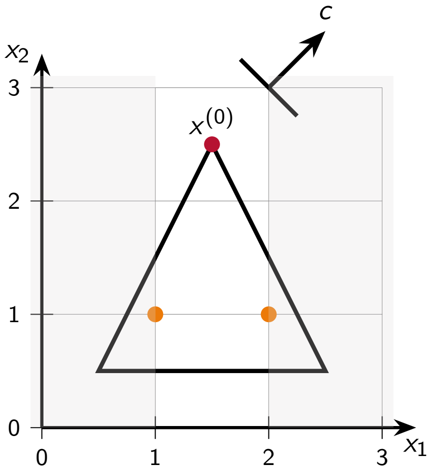
3. two subproblems



Example: Branch & Bound Tree





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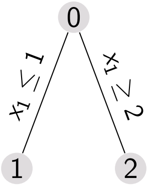

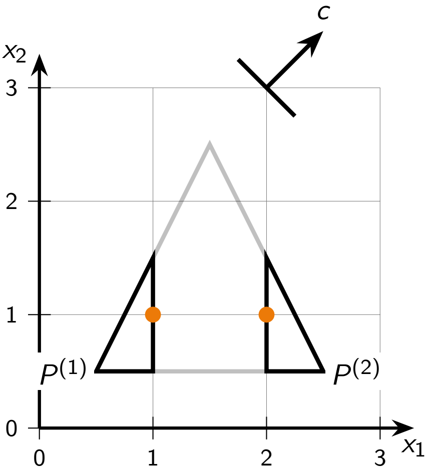

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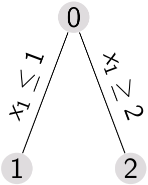

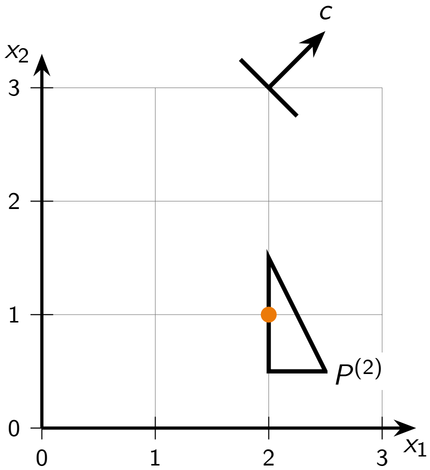

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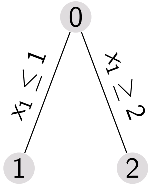

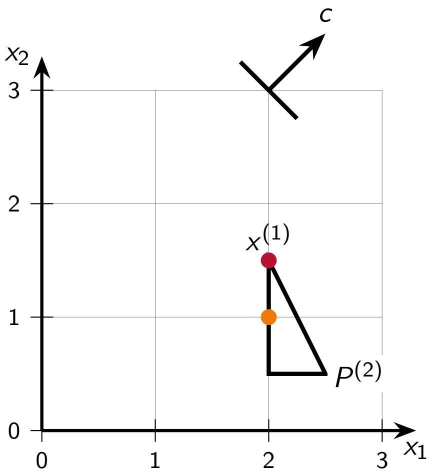

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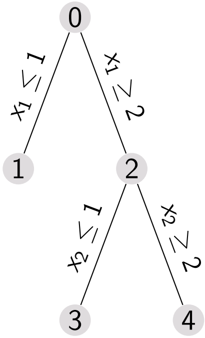

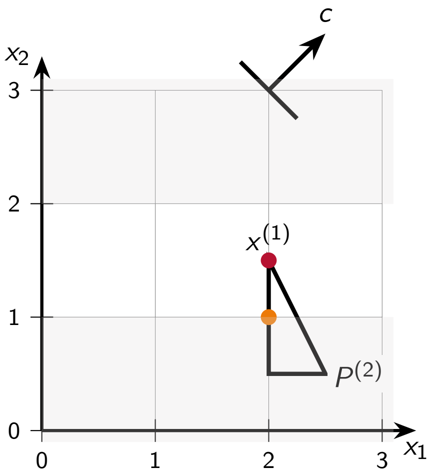

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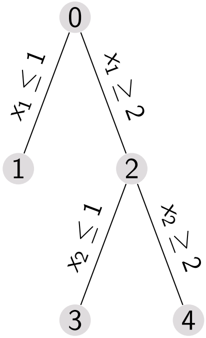

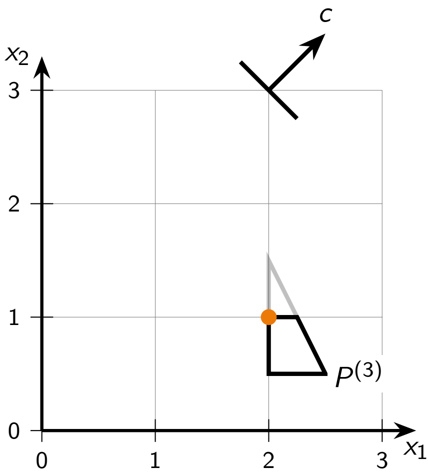

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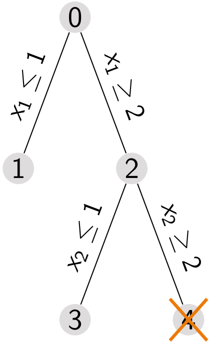

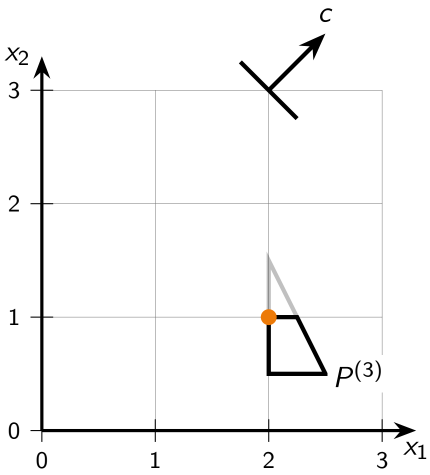

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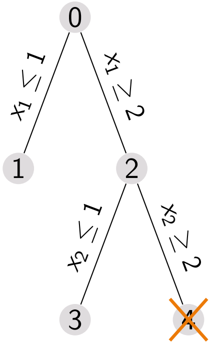

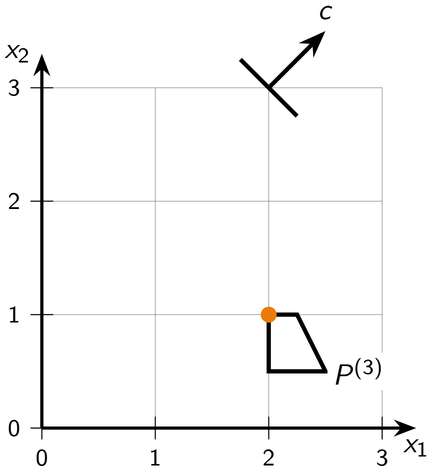

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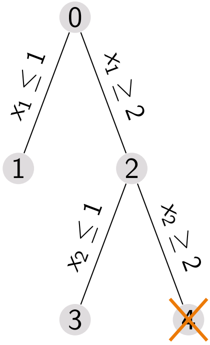

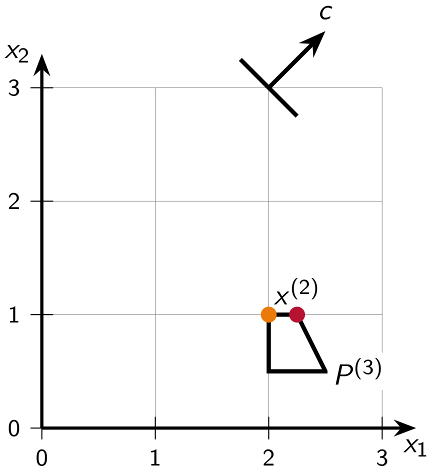

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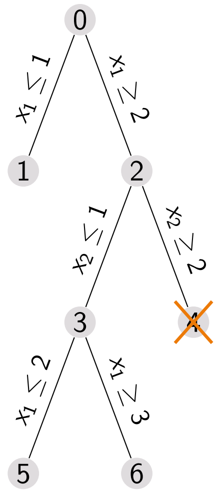

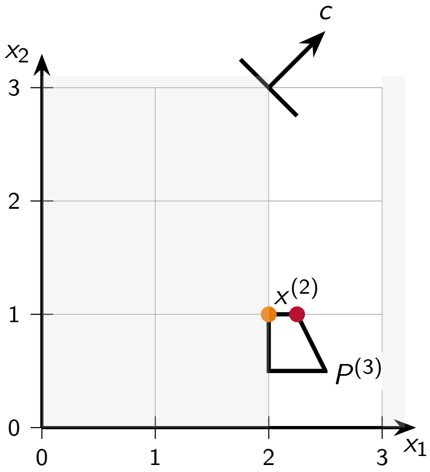

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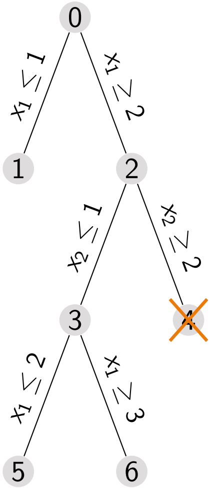

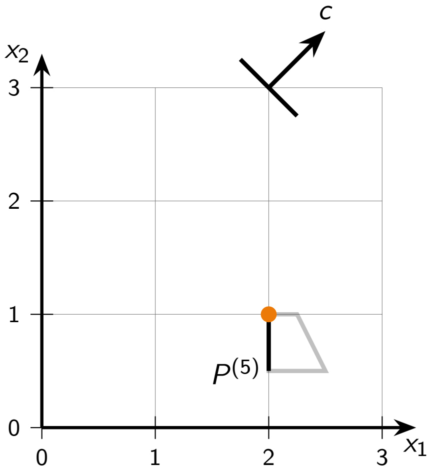

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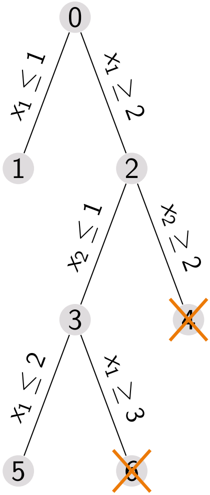

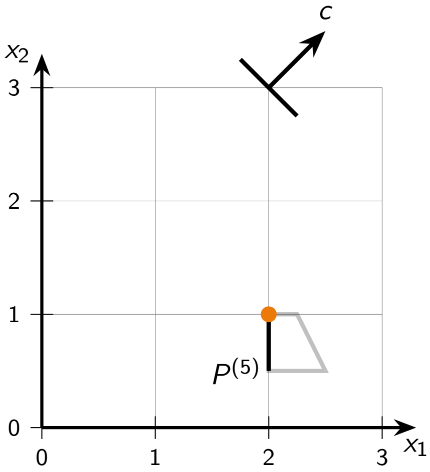

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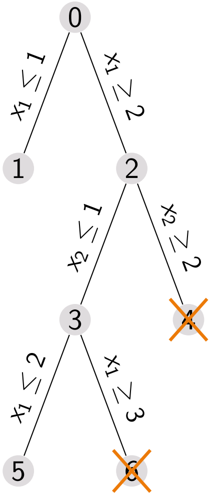

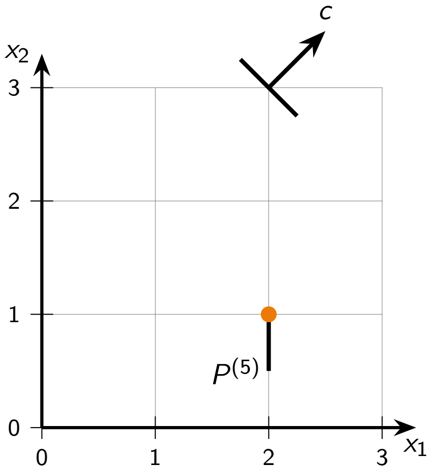

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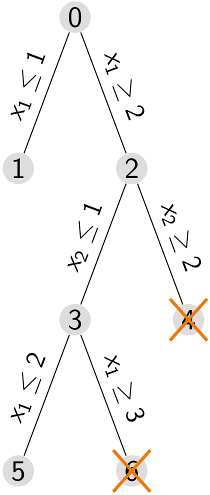

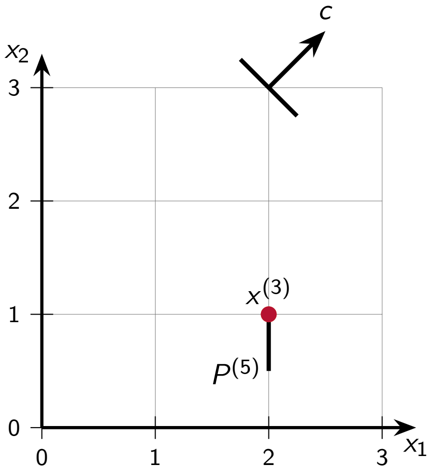

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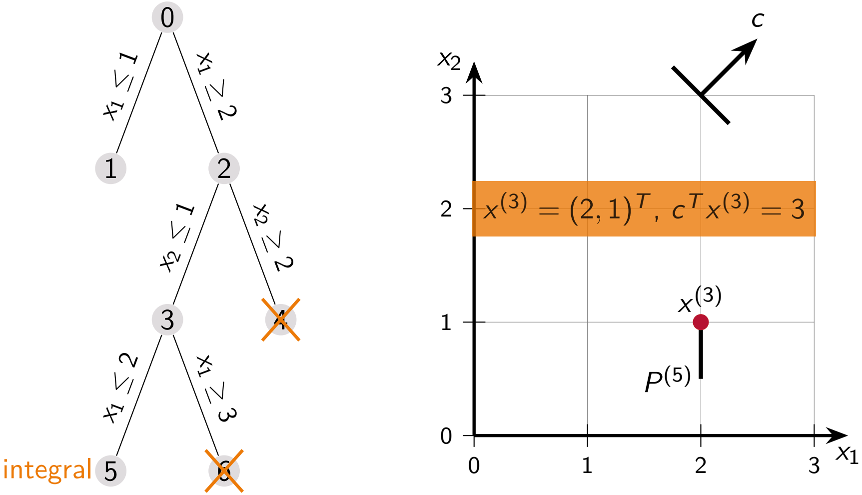

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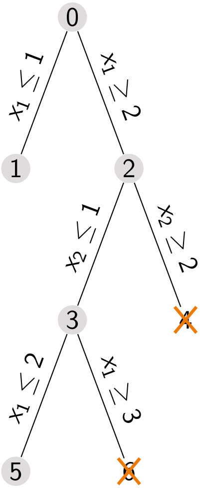
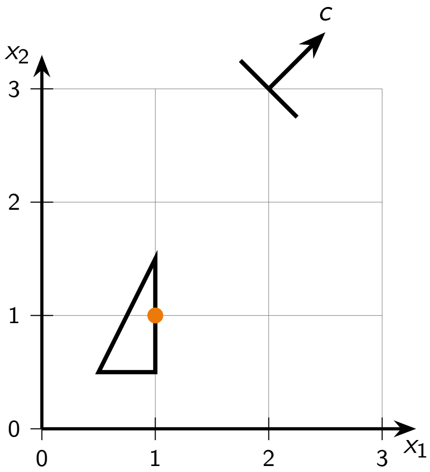
- idea: prune nodes
  - for each node: determine upper bound S
  - $S \leq c^T x^*$ for best known solution x^*
    $\rightarrow$ prune node
- for min-ILP:
  - upper bound U for the problem (current best solution)
  - lower bound S for each node
  - if $S \geq U \rightarrow$ prune node

Example: Bounding and Pruning



$c^T x^* = 3$



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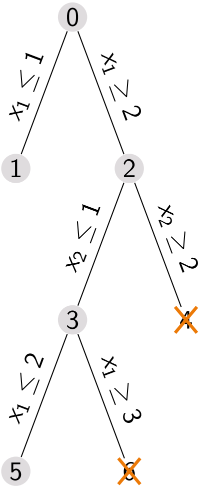

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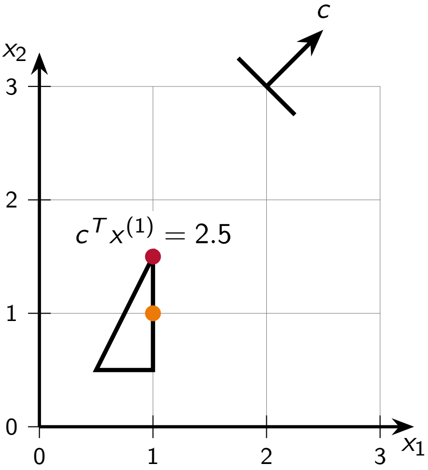

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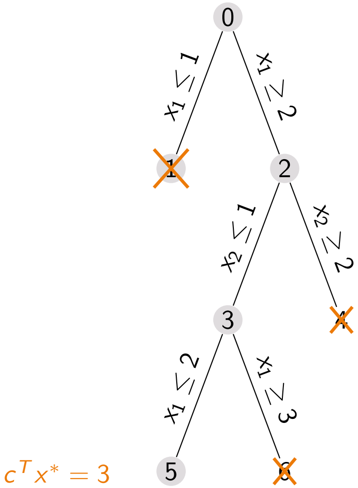

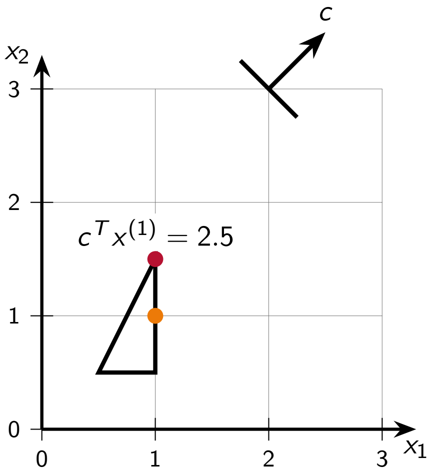

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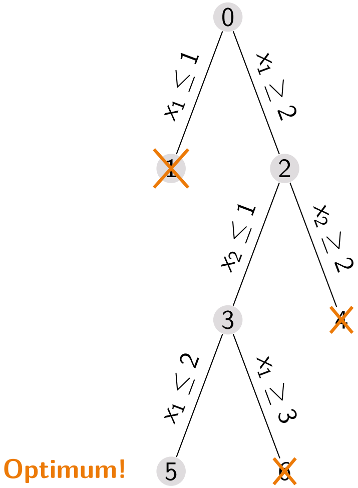

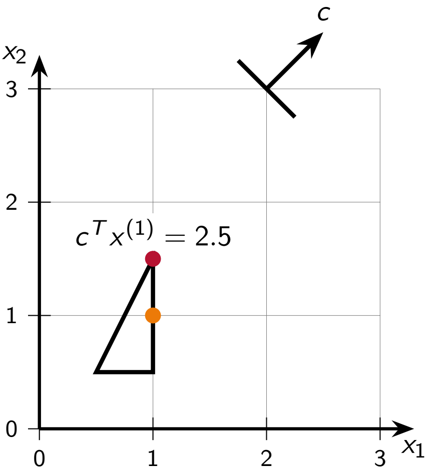

Interactive: Knapsack

Problem: Knapsack	
Given:	$n \in \mathbb{N}$ items with
	values $v_i \in \mathbb{N}$ and
	weights $w_i \in \mathbb{N}$
	knapsack capacity $W \in \mathbb{N}$
Task:	Maximize the total value of the knapsack
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$$
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\text{s.t.} & \sum_{i=1}^{n} w_i x_i & \leq & W \\
 & x & \in & \{0,1\}^n
\end{array}
$$

Interactive: Knapsack (Multiple Copies)

Problem: Knapsack (Multiple Copies)	
Given:	$n \in \mathbb{N}$ items with
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$$\begin{array}{lrcll} \max & \sum_{i=1}^{n} v_i y_i & & & \\ \text{s.t.} & \sum_{i=1}^{n} w_i y_i & \leq & W & \\ & y_i & \leq & p_i & \text{for all } i \in [n] \\ & y & \in & \mathbb{N}^n & \end{array}$$