Approximation Algorithms

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Primal-dual

Knapsack cover

Knapsack cover

Input: demand D, items $I = \{1, \ldots, n\}$ with values v_i and sizes $s_i \in \mathbb{R}_+$.

Task: find a subset $S \subseteq I$ with total value at least D minimizing $\sum_{i \in S} s_i$.

Primal:

$$\begin{aligned} \min \quad & \sum_{i \in I} s_i x_i \\ \text{s.t.} \quad & \sum_{i \in I} v_i \; x_i \geq D \\ & x_i \geq 0 \qquad \forall i \in I \end{aligned}$$

Dual:

$$\begin{aligned} \max \quad & \sum_{A \subseteq I} D_A y_A \\ \text{s.t.} \quad & \sum_{A \subseteq I : i \notin A} v_i^A y_A \leq s_i \qquad \forall\, i \in I \\ & y_A \geq 0 \qquad \forall A \subseteq I \end{aligned}$$

$$\boxed{v_i^A = \min\{v_i, D_A\}}$$

Primal-dual algorithm

Important

Start increasing the “small object” dual variable. Or, increase greadily the dual variable with highest coefficient in the objective.

- Here: $y_\emptyset$.
- Until the dual constraint is tight for some $i \in I$, add i to the set S.
- Increase y_S.

Theorem

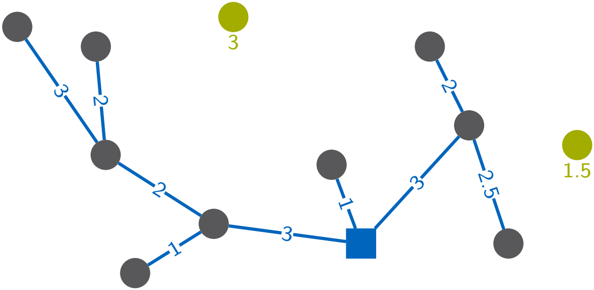
The algorithm is a 2 approximation for knapsack cover.

Primal-dual: Prize-collecting Steiner Tree

Prize-collecting steiner tree

Input: graph $G = (V \cup \{r\}, E)$, distances* $d : E \to \mathbb{R}_+$, penalties $\pi : V \to \mathbb{R}_+$

Task: find $U \subseteq V$ and a tree T spanning $U \cup \{r\}$ minimizing $\sum_{e \in T} d_e + \sum_{v \in V \setminus U} \pi_v$



*w.l.o.g.: G is complete and d is metric

Before

Deterministic LP rounding

3-approximation

Randomized LP rounding

2.54-approximation (and derandomization)

Integrality gap

No LP based $(2 - \varepsilon)$-approximation.

Today

2-approximation through primal-dual

New LP relaxation

variables:

$$x_e = 1 \;\Leftrightarrow\; e \in T$$

$$z_X = 1 \;\Leftrightarrow\; X = V \setminus T$$

Primal:

$$\begin{aligned}
\min \quad & \sum_{e \in E} d_e x_e + \sum_{X \subseteq V} \pi(X) z_X \\
\text{s.t.} \quad & \sum_{e \in \delta(S)} x_e + \sum_{X : X \supseteq S} z_X \geq 1 && \forall\, S \subseteq V,\; S \not\subseteq X \\
& x_e \geq 0 && \forall\, e \in E \\
& z_X \geq 0 && \forall\, X \subseteq V
\end{aligned}$$

Dual:

$$\begin{aligned}
\max \quad & \sum_{S \subseteq V} y_S \\
\text{s.t.} \quad & \sum_{S : e \in \delta(S)} y_S \leq d_e && \forall\, e \in E
\end{aligned}$$

The primal-dual algorithm

Definition

$p(X, y, \pi) = \pi(X) - \sum_{S:S\subseteq X} y_S$ is the potential of X.

Idea 1

If the potential of a set X is zero, the penalties of not selecting X can be "payed" from the dual.

Idea 2

Maintain a dual feasible solution y and an infeasible primal solution F (a forest).

(In)active components

A connected component of F is inactive if either: 1) it contains the root, or 2) its potential is zero. It is active otherwise.

The primal-dual algorithm

Algorithm

1. Initialize
   - $y := 0$; $F := \emptyset$
2. While (not all connected components of $(V \cup \{r\}, F)$ are inactive):
   - $\mathcal{C} :=$ set of all active connected components of (V, F).

   do Increase y_C for all $C \in \mathcal{C}$ equally until either for an edge $e \in E$ the dual constraint is tight, or for some set $C \in \mathcal{C}$ the dual constraint is tight (potential is zero).

   if for some $C \in \mathcal{C}$, $p(C, y, \pi) = 0$:
   C is inactive, remove C from $\mathcal{C}$.

   else ($e \in E$ has tight dual ineq.):
   $F := F \cup \{e\}$
3. $T :=$ connected component containing the root r.
   In reverse order of addition to F consider all $e \in F'$:
   - For V_e the vertices that are disconnected from r by removing e:

   if $P(V_e, y, \pi) = 0$:
   Remove e and the subtree rooted at the end of e.

The primal-dual algorithm

Theorem

The primal dual algorithm is a 2-approximation for prize-collecting steiner tree.

Lemma

At any iteration of the algorithm, with T the final solution, X all vertices not spanned by T, and $\mathcal{C}$ the active components in that iteration:

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap T| \; + \; |\{C \in \mathcal{C} \; : \; C \subseteq X\}| \leq 2|\mathcal{C}| \, .$$