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Approximation Algorithms

Notes on Lecture 14: Rounding of semidefinite programs (20.06.2019)

Algorithm to check positive-semidefiniteness of a matrix M and get its $M = U^T U$ decomposition:

1. Let α be the top left corner of M and $\boldsymbol{q}^T$ its first row. In other words,
$$M = \begin{bmatrix} \alpha & \boldsymbol{q}^T \\ \boldsymbol{q} & N \end{bmatrix}.$$
2. If $\alpha < 0$, report that M is not positive semidefinite (psd).
3. If $\alpha = 0$, then by Lemma 1 (see below) we know that $\boldsymbol{q} = \mathbf{0}$ or the matrix is not positive semidefinite. We recurse on N. If we get a decompositon $N = V^T V$, we add 0 as a first coordinate to each vector $v_i \in V$, add a zero vector $\mathbf{0}$ and return that as our decomposition U.
4. If $\alpha > 0$, we observe (by simple matrix multiplication) that the following equation is true:
$$M = \begin{bmatrix} \sqrt{\alpha} & \mathbf{0}^T \\ \frac{1}{\sqrt{\alpha}}\boldsymbol{q} & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & N - \frac{1}{\alpha}\boldsymbol{q}\boldsymbol{q}^T \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & \frac{1}{\sqrt{\alpha}}\boldsymbol{q}^T \\ \mathbf{0} & I_{n-1} \end{bmatrix}$$
We will call the first matrix in the product as A and the second as R, thus getting
$$M = A^T R A.$$
We also denote $N - \frac{1}{\alpha}\boldsymbol{q}\boldsymbol{q}^T$ by N' (as it is *almost* the same as N).
5. We call our decomposition algorithm recursively on the $(n-1) \times (n-1)$ matrix N'. By Lemma 2 (see below) we have that the recursive matrix is positive semidefinite if and only if M is positive semidefinite.
6. If the recursive procedure reports that the matrix is not psd, we also report that M is not psd.
7. Otherwise, we take the recursive decomposition of N' into $V^T V$ and create a decomposition of R. R can be decomposed as $R = V'^T V'$ where V'^T is an $n \times n$ matrix that has $n-1$ rows of the form $(0, \boldsymbol{v})$ (coming from the rows of V^T) and one row $(1, 0, 0, \ldots, 0)$ (for the top left 1 that is present in R).

8. Finally, we set U to be $U^T = A^T V'^T$ and return it as our solution U.

The algorithm finishes in polynomial time. The correctness of the algorithm comes from the following two lemmas:

Lemma 1. *If a matrix M has the form $M = \begin{bmatrix} \alpha & \boldsymbol{q}^T \\ \boldsymbol{q} & N \end{bmatrix}$ with $\alpha > 0$ and if M is psd, then also the matrix $N - \frac{1}{\alpha}\boldsymbol{q}\boldsymbol{q}^T$ is psd.*

Proof. Recall the decomposition $M = A^T R A$. Looking at the matrix A, observe that it has rank n (it is triangular with a non-zero diagonal), and therefore it is invertible. Invertibility means that given any vector $\boldsymbol{y}$ we can compute a vector $\boldsymbol{z}$ such that $A\boldsymbol{z} = \boldsymbol{y}$.

We move on with the proof. Our goal is to show that for any vector $x' \in \mathbb{R}^{n-1}$ it is true that $x'^T N' x' \geq 0$. To show this, we create a vector $(0, \boldsymbol{x'})$ and then we compute $\boldsymbol{z}$ as the inverse of $(0, \boldsymbol{x'})$ for matrix A.

Due to our construction, we have

$$\boldsymbol{z}^T M \boldsymbol{z} = \boldsymbol{z}^T A^T R A \boldsymbol{z} = (0, \boldsymbol{x'})^T R (0, \boldsymbol{x'}) = x'^T N' x'.$$

The last equality in that chain follows from the fact that the vector $(0, \boldsymbol{x'})$ has 0 as its first coordinate and therefore we can move from R to N' by ignoring the first row and column.

Finally, from the fact that M was positive semidefinite, we have that $\boldsymbol{z}^T M \boldsymbol{z} \geq 0$ for any choice of $\boldsymbol{z}$. Combining the equalities with the last fact, we get $x'^T N' x' \geq 0$, which concludes the proof. □

Lemma 2. *If a matrix M has the form*

$$M = \begin{bmatrix} 0 & \boldsymbol{q}^T \\ \boldsymbol{q} & N \end{bmatrix}$$

and if M is psd, then $\boldsymbol{q} = \boldsymbol{0}$.

Proof. Let us consider a vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$. We will focus on the first coordinate, shortcutting the remaining coordinates of $\boldsymbol{v}$ as $\boldsymbol{v} = (v_1, \boldsymbol{v'})$.

Since M is given to be positive semidefinite, we know that $\boldsymbol{v}^T M \boldsymbol{v} \geq 0$ for any choice of $\boldsymbol{v}$. We now expand the product using our notation so far, getting:

$$\boldsymbol{v}^T M \boldsymbol{v} = (v_1, \boldsymbol{v'})^T \cdot \begin{bmatrix} 0 & \boldsymbol{q}^T \\ \boldsymbol{q} & N \end{bmatrix} \cdot (v_1, \boldsymbol{v'}) = 2v_1 \langle \boldsymbol{q}, \boldsymbol{v'} \rangle + \boldsymbol{v'}^T N \boldsymbol{v'}.$$

Now, we recall that the right-hand side of the equality above must be non-negative for all choices of $\boldsymbol{v}$. This includes the choices where v_1 is an arbitrarily large negative integer (or positive integer in the case that $\langle \boldsymbol{q}, \boldsymbol{v'} \rangle < 0$). The only way that can be the case if $2v_1 \langle \boldsymbol{q}, \boldsymbol{v'} \rangle = 0$. This term must be zero for all choices of $\boldsymbol{v'}$, and the only way this can be true is if the vector $\boldsymbol{q}$ is zero in all coordinates. □