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Approximation Algorithms

Notes on Lecture 10: Rounding Price collecting steiner tree (23.05.2019)

Claim 1 (Deterministic rounding Claim 1)**.**

$$d(T) \leq \frac{2}{\alpha} \sum_{e \in E} d(e) x^*_e \,.$$

Proof. $\frac{1}{\alpha} x^*$ is a feasible solution for the following LP (the Steiner tree LP)

$$\begin{aligned} \min \quad & \sum_{e \in E} d(e) x_e & & \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 1 & & \forall S \subseteq V : S \cap U \neq \emptyset \\ & x_e \geq 0 & & \forall e \in E \end{aligned} \tag{LP}$$

Note that E is still a set of edges on $V \cup \{r\}$. (LP) is the LP relaxation of the Steiner tree problem (where $U \cup \{r\}$ are the terminals).

Definition 1 (Steiner tree problem)**.**

Input: *Graph $G = (V, E)$, distances $d(e)$ for all $e \in E$, set of terminals $R \subseteq V$.*

Task: *Find a tree T^* that connects all terminals.*

Objective: *Minimize the total distance of the tree $\sum_{e \in T^*} d(e)$.*

The MST T on $U \cup \{r\}$ is a 2-approximation of (LP). Here, we only prove that T is a 2-approximation of the Steiner tree problem.

Consider an optimal Steiner tree T^*. Traverse the boundary of T^* to obtain a tour that uses each edge in T^* exactly twice. The cost of this tour is $2 \sum_{e \in T^*} d(e)$. Now shortcut the tour to obtain a tour of only the terminals $U \cup \{r\}$. Remove one edge from this tour to obtain a path. Since this path is a tree, its total distance is at least that of the MST. Since we assumed, w.l.o.g. that the distances in the graph are metric, the shortcuts only decreased the distance and the MST has total distance at most $2 \sum_{e \in T^*} d(e)$. □

Claim 2 (Deterministic rounding Claim 2)**.**

$$\pi(V \setminus U) \leq \frac{1}{1 - \alpha} \sum_{v \in V} \pi(v)(1 - y^*_v) \,.$$

Proof. For all $v \in V \setminus U$ we have that $y_v^* \leq \alpha$ by definition, thus $1 - y_v^* > 1 - \alpha$ and $\frac{1-y_v^*}{1-\alpha} > 1$. ☐

Proof: Deterministic rounding theorem. $d(T) + \pi(V \setminus U) \leq \max\{\frac{2}{\alpha}, \frac{1}{1-\alpha}\}\mathrm{OPT} \leq 3\mathrm{OPT}$ (the last inequality follows from setting $\alpha = \frac{2}{3}$). ☐

Claim 3 (Randomized rounding Claim 1r)**.**

$$\mathbb{E}[d(T)] \leq (\frac{2}{1-\gamma} \ln \frac{1}{\gamma}) \sum_{e \in E} d(e) x_e^* \,.$$

Proof.

$$\begin{aligned}
\mathbb{E}[d(T)] &\leq \mathbb{E}\left[\frac{2}{\alpha} \sum_{e \in E} d(e) x_e^*\right] && \text{(By Claim 1)} \\
&= \mathbb{E}\left[\frac{2}{\alpha}\right] \sum_{e \in E} d(e) x_e^* \\
&= \left(\int_\gamma^1 \frac{2}{z} \frac{1}{1-\gamma} dz\right) \sum_{e \in E} d(e) x_e^* && \text{(Prob. density funct. of } \alpha \text{ is } \frac{1}{1-\gamma}) \\
&= \left[\frac{2}{1-\gamma} \ln z\right]_\gamma^1 \sum_{e \in E} d(e) x_e^* \\
&= (\frac{2}{1-\gamma} \ln \frac{1}{\gamma}) \sum_{e \in E} d(e) x_e^* \,.
\end{aligned}$$ ☐

Claim 4 (Randomized rounding Claim 2r)**.**

$$\mathbb{E}[\pi(V \setminus U)] \leq \frac{1}{1-\gamma} \sum_{v \in V} \pi(v)(1 - y_v^*) \,.$$

Proof. If $y_v^* \leq \gamma$ then $\mathbb{P}[v \in V \setminus U] = 1 \leq \frac{1-y_v^*}{1-\gamma}$.
If $y_v^* \geq \gamma$ then $\mathbb{P}[v \in V \setminus U] = \mathbb{P}[\alpha \geq y_v^*] = \frac{1-y_v^*}{1-\gamma}$. ☐