Approximation Algorithms

Ruben Hoeksma (University of Bremen)

May 23rd, 2019

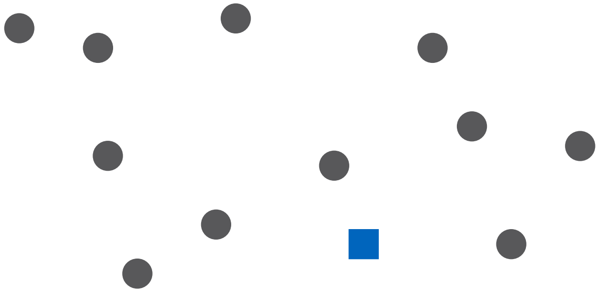
LP Rounding: Prize-collecting Steiner Tree

Prize-collecting Steiner Tree

Input: graph $G = (V \cup \{r\}, E)$, distances $d : E \to \mathbb{R}_+$,
penalties $\pi : V \to \mathbb{R}_+$.

Task: find $U \subseteq V$ and a tree T spanning $U \cup \{r\}$.

Objective: minimize $\sum_{e \in T} d(e) + \sum_{v \in V \setminus U} \pi(v)$.

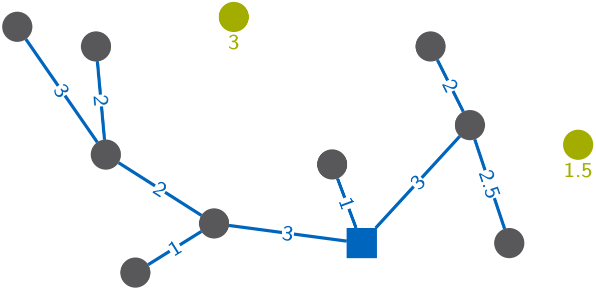


Prize-collecting Steiner Tree

Input: graph $G = (V \cup \{r\}, E)$, distances $d : E \to \mathbb{R}_+$, penalties $\pi : V \to \mathbb{R}_+$.

Task: find $U \subseteq V$ and a tree T spanning $U \cup \{r\}$.

Objective: minimize $\sum_{e \in T} d(e) + \sum_{v \in V \setminus U} \pi(v)$.

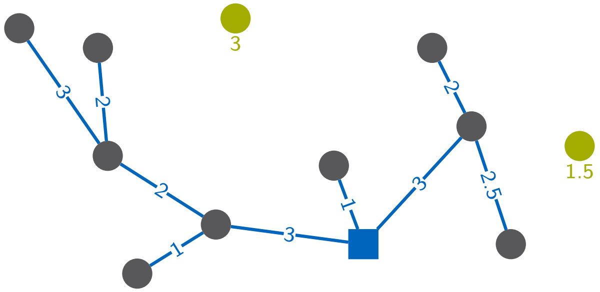


Prize-collecting Steiner Tree

Input: graph $G = (V \cup \{r\}, E)$, distances* $d : E \to \mathbb{R}_+$, penalties $\pi : V \to \mathbb{R}_+$.

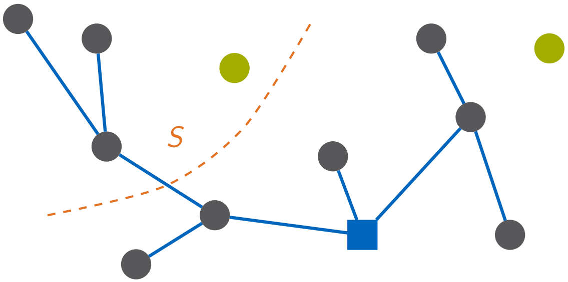
Task: find $U \subseteq V$ and a tree T spanning $U \cup \{r\}$.

Objective: minimize $\sum_{e \in T} d(e) + \sum_{v \in V \setminus U} \pi(v)$.



*w.l.o.g.: G is complete and d is metric

LP relaxation



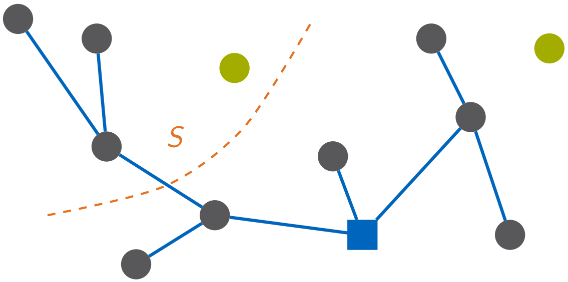
variables:

$x(e) = 1 \;\Leftrightarrow\; e \in T$

$y(v) = 1 \;\Leftrightarrow\; v \in U$

$$\begin{aligned}
\min \quad & \sum_{e \in E} d(e)x(e) \;+\; \sum_{v \in V} \pi(v)(1 - y(v)) & \\
\text{s.t.} \quad & \sum_{e \in \delta(S)} x(e) \;\geq\; y(v) & \forall\, S \subseteq V,\; \forall\, v \in S \\
& x(e) \;\in \{0,1\} & \forall\, e \in E \\
& y(v) \;\in \{0,1\} & \forall\, v \in V
\end{aligned}$$

LP relaxation



variables:

$x(e) = 1 \;\Leftrightarrow\; e \in T$

$y(v) = 1 \;\Leftrightarrow\; v \in U$

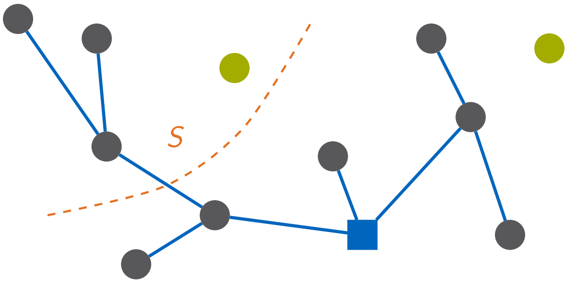
$$\min \sum_{e \in E} d(e)x(e) + \sum_{v \in V} \pi(v)(1 - y(v))$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x(e) \geq y(v) \qquad \forall\, S \subseteq V,\ \forall\, v \in S$$

$$x(e) \in \{0,1\} \qquad \forall\, e \in E$$

$$y(v) \in \{0,1\} \qquad \forall\, v \in V$$

LP relaxation



variables:

$x(e) = 1 \iff e \in T$

$y(v) = 1 \iff v \in U$

$$\min \sum_{e \in E} d(e)x(e) + \sum_{v \in V} \pi(v)(1 - y(v))$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x(e) \geq y(v) \qquad \forall\, S \subseteq V,\ \forall\, v \in S$$

$$x(e) \geq 0 \qquad \forall\, e \in E$$

$$1 \geq y(v) \geq 0 \qquad \forall\, v \in V$$

A 3-approximation algorithm

Algorithm D

1. Compute optimal solution (x^*, y^*) to LP.
2. Let $U := \{v \in V : y^*(v) \geq \alpha\}$.
3. Let T be minimum spanning tree on $U \cup \{r\}$.
4. Return T and U.

A 3-approximation algorithm

Algorithm D

1. Compute optimal solution (x^*, y^*) to LP.
2. Let $U := \{v \in V \;:\; y^*(v) \geq \alpha\}$.
3. Let T be minimum spanning tree on $U \cup \{r\}$.
4. Return T and U.

Claim 1: $$d(T) \;\leq\; \frac{2}{\alpha} \cdot \sum_{e \in E} d(e) x^*(e)$$

Claim 2: $$\pi(V \setminus U) \;\leq\; \frac{1}{1-\alpha} \cdot \sum_{v \in V} \pi(v)(1 - y^*(v))$$

A 3-approximation algorithm

Algorithm D

1. Compute optimal solution (x^*, y^*) to LP.
2. Let $U := \{v \in V \,:\, y^*(v) \geq \alpha\}$.
3. Let T be minimum spanning tree on $U \cup \{r\}$.
4. Return T and U.

Claim 1: $d(T) \leq \frac{2}{\alpha} \cdot \sum_{e \in E} d(e) x^*(e)$

Claim 2: $\pi(V \setminus U) \leq \frac{1}{1-\alpha} \cdot \sum_{v \in V} \pi(v)(1 - y^*(v))$

Theorem

Algorithm D is a 3-approximation algorithm for Prize-collecting Steiner Tree (when setting $\alpha = 2/3$).

Improvement with randomization

Algorithm B (Best of Many)

1. Compute optimal solution (x^*, y^*) to LP.
2. Run Algorithm D for every $\alpha \in \{y^*(v) : v \in V\}$.
3. Return best solution found.

Improvement with randomization

Algorithm B (Best of Many)

1. Compute optimal solution (x^*, y^*) to LP.
2. Run Algorithm D for every $\alpha \in \{y^*(v) : v \in V\}$.
3. Return best solution found.

Algorithm R (Randomized rounding)
Choose α uniformly at random from $[\gamma, 1]$ and run Algorithm D.

Observation: Algorithm B is at least as good as Algorithm R.
(randomized analysis)

Improvement with randomization

Theorem

Algorithm R is a randomized 2.54-approximation algorithm for Prize-collecting steiner tree (when setting $\gamma = e^{-1/2}$).

Claim 1r: $\mathbb{E}[d(T)] \le \left(\frac{2}{1-\gamma} \ln \frac{1}{\gamma}\right) \cdot \sum_{e \in E} d(e) x^*(e)$

Claim 2r: $\mathbb{E}[\pi(V \setminus U)] \le \frac{1}{1-\gamma} \cdot \sum_{v \in V} \pi(v)(1 - y^*(v))$

Improvement with randomization

Theorem

Algorithm R is a randomized 2.54-approximation algorithm for Prize-collecting steiner tree (when setting $\gamma = e^{-1/2}$).

Derandomization

Try all $\gamma = y^*(v)$. These are n possibilities and find all possible outcomes for any γ.

Improvement with randomization

Theorem

Algorithm R is a randomized 2.54-approximation algorithm for Prize-collecting steiner tree (when setting $\gamma = e^{-1/2}$).

Derandomization

Try all $\gamma = y^*(v)$. These are n possibilities and find all possible outcomes for any γ.

Integrality gap

No algorithm using the LP relaxation as a lower bound can have a $(2 - \varepsilon)$ approximation factor.