Approximation Algorithms

Martin Böhm (University of Bremen)

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Linear programming and duality

LP duals - general

Idea

Create good bounds on the optimal solution.

Example:

$$\begin{array}{rrcrcl} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & 4x_1 & + & 8x_2 & \leq & 12 \\ & 2x_1 & + & x_2 & \leq & 3 \\ & 3x_1 & + & 2x_2 & \leq & 4 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

$$\begin{array}{rrcrcrcl} \min & 12y_1 & + & 3y_2 & + & 4y_3 & & \\ \text{s.t.} & 4y_1 & + & 2y_2 & + & 3y_3 & \geq & 2 \\ & 8y_1 & + & y_2 & + & 2y_3 & \geq & 3 \\ & & & & & & & \\ & y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

LP duals - general

Idea

For primal

$$\min\{c^t x \mid Ax \geq b,\ x \geq 0\}\,,$$

its dual is

$$\max\{b^t y \mid A^t y \leq c,\ y \geq 0\}\,.$$

$$\begin{array}{rrcl}
\min & c^t x & & \\
\text{s.t.} & a_i x & \geq & b_i \text{ for } i \in I_1 \\
& a_i x & = & b_i \text{ for } i \in I_2 \\
& x_j & \geq & 0 \text{ for } j \in J_1 \\
& x_j & \in & \mathbb{R} \text{ for } j \in J_2
\end{array}
\qquad
\begin{array}{rrcl}
\max & b^t y & & \\
\text{s.t.} & y_i & \geq & 0 \text{ for } i \in I_1 \\
& y_i & \in & \mathbb{R} \text{ for } i \in I_2 \\
& A_j y & \leq & c_j \text{ for } j \in J_1 \\
& A_j y & = & c_j \text{ for } j \in J_2
\end{array}$$

LP Duals - Theorems

Theorem - Weak duality

Let $P = \min\{c^t x \mid Ax \geq b,\ x \geq 0\}$ and $D = \max\{b^t y \mid A^t y \leq c,\ y \geq 0\}$. If x is a feasible solution for P and y is a feasible solution for D, then $c^t x \geq b^t y$.

Proof.

$$
\begin{aligned}
c^t x &= x^t c \\
&\geq x^T (A^T y) \\
&= (Ax)^T y \\
&\geq b^T y
\end{aligned}
$$

□

LP Duals - Theorems

Theorem - Strong duality

If P and D are a primal-dual pair of LPs, then exactly one of the following holds:

1. Both P and D are infeasible.
2. P is unbounded and D is infeasible.
3. D is unbounded and P is infeasible.
4. Both P and D are feasible and there are optimal solutions x^*, y^* such that $c^T x^* = b^T y^*$.

Example: Scheduling with release dates

$1|r_j|\sum C_j$

Input: n jobs with processing time p_j and release date r_j

Task: Find non-preemptive schedule of the jobs (no job j scheduled before r_j)

Objective: Minimize sum of completion times $\sum_{j\in[n]} C_j$

Relaxation

Preemptive schedules: we are allowed to stop (preempt) the processing of jobs and resume them later.

Optimal preemptive solution

Schedule according to shortest remaining processing time (SRPT) rule.

$1|r_j|\sum C_j$

C_j^P: preemptive optimal completion time of job j.

Lemma

$$\sum_{j=1}^{n} C_j^P \leq \mathrm{OPT}$$

Idea

Schedule non-preemptively in order of C_j^P.

C_j^N: non-preemptive optimal completion time of job j.

Theorem

$\sum_{j=1}^{n} C_j^N \leq 2\,\mathrm{OPT}$.

$1|r_j|\sum w_j C_j$

Weighted version: jobs have weights w_j.

Objective: $\sum_{j=1}^n w_j C_j$

NP-hard to solve preemptive problem. $\Rightarrow$ Use a different relaxation.

LP

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} w_j C_j & \\
\text{s.t.} \quad & C_j \geq r_j + p_j & \forall j \\
& \sum_{j \in S} pjC_j \geq \frac{1}{2} \left(\sum_{j \in S} p_j \right)^2 & \forall S \subseteq [n] \\
& C_j \geq 0 & \forall j
\end{aligned}
$$

$1|r_j|\sum w_j C_j$

Theorem

Let C^* be an optimal LP solution, then scheduling in order of completion times C_j^* is a 3-approximation for minimizing weighted sum of completion times with release dates on a single machine.