- Move only last job on a machine
- Always move to the machine with smallest load.

Look at the last job to finish j^*

$$C_{max} = C_{j^*} = s_{j^*} + p_{j^*}$$

$$s_{j^*} \leq L_i \quad \forall i \in [m]$$

$$\Rightarrow s_{j^*} \leq \frac{\sum_{j \in [n]} p_j - p_{j^*}}{m}$$

$$\sum_{j \in [n]} p_j \leq C_{max}(OPT)$$

$$p_j \leq C_{max}(OPT) \quad \forall j$$

$$\Rightarrow C_{max} = s_{j^*} + p_{j^*} = \frac{\sum_{j \in [n]} p_j}{m} + \left(1 - \frac{1}{m}\right) p_{j^*} \leq \left(2 - \frac{1}{m}\right) C_{max}(OPT) \quad \square$$

Poly time?

Let C_{min} be the load of the least loaded machine.

Claim: C_{min} does not decrease while the algorithm runs.

Load is always moved to the least loaded machine.

Claim: Each job moves at most once.

Proof: Suppose not and let job j be a job that moves at least twice.

Suppose it moves from machine $i \to i' \to i''$.

When it moves it is the last job on the machine.

Let $L_{i'}$ be the load of i' when j move there

Then $L_{i'} = C_{min}$ at that time. When j moves from i' to i'' the load on i'' is C_{min} however that cannot be less than $L_{i'}$ so is not an improving move. $\lightning$

Thm local search is a 5-approx. for k-Median.

Proof.

S: local optimum $|S| = k$ $A_i = d(i, S)$ $A = \sum A_i$

S^*: optimal solution $|S^*| = k$ $OPT_i = d(i, S^*)$ $OPT = \sum OPT_i$

Goal: Show that there are k swaps that imply the bound, by being non-improving.

Map $\sigma : S^* \to S$ such that $\sigma(i^*) \in S$ and $\sigma(i^*)$ is closest to i^* i.e. $d(i^*, \sigma(i^*)) \leq d(i^*, i)$ $\forall i \in S$.

Let:

O be the set of nodes $i \in S$ s.t. there is exactly one $i^* : \sigma(i^*) = i$

Z " " " " " " " " " no $i^* : \sigma(i^*) = i$

T " " " " " " " " " two or more $i^* : \sigma(i^*) = i$

O^* be the set of nodes $i^* \in S^*$ s.t. there is no other $j^* \in S^*$: $\sigma(i^*) = \sigma(j^*)$

R^* be $S^* \setminus O^*$

Define the following k swaps: (Set $\mathcal{S}$)

- $(i^*, \sigma(i^*))$ $\forall i^* \in O^*$
- $\forall i \in Z$ and $i^* \in R^*$ create arbitrary pairs s.t. each $i \in Z$ has at most two pairs.

The latter can be done since $|R^*| = |Z| + |T|$

and $|T| \leq \frac{1}{2}|R^*| \Rightarrow |Z| \geq \frac{1}{2}|R^*|$

fact: $\forall (i^*, i)$ and $\hat{i}^* \in S^*$: $\hat{i}^* \neq i^*$, we have $\sigma(\hat{i}^*) \neq i$.

Intuition: $O \to O^*$ swaps are nice

$Z/T \to R^*$ swaps are problematic (no $T \to R^*$ swaps)

Analysis:

Let $\varphi : V \to S$ map vertices to nearest facility in S.

" $\varphi^* : V \to S^*$ " " " " " " S^*.

Let $N^*(i^*) = \{ j \in V \mid \varphi^*(j) = i^* \}$

Let $N(i) = \{ j \in V \mid \varphi(j) = i \}$

Lem $\forall\ (i^*, i) \in \mathcal{P}$

$$0 \le \sum_{j \in V} d(j, S + i^* - i) - \sum_{j \in V} d(j, S) \le \sum_{j \in N^*(i^*)} (OPT_j - A_j) + \sum_{j \in N(i)} 2\, OPT_j$$

Proof. Create an assignment for $S + i^* - i$

$\forall\ j \in N^*(i^*)$ $\quad j \to i^*$

$\forall\ j \in N(i) \setminus N^*(i^*)$ $\quad$ Let $\hat{i}^* = \varphi^*(j) \neq i^*$.

$j \to \hat{i} = \sigma(\hat{i}^*)$. By **fact** $\sigma(\hat{i}^*) \neq i$, so this assignment is valid. Also $\hat{i}$ is the closest facility in S to $\hat{i}^*$.

$\forall j \in N^*(i^*)$ change in cost is $OPT_j - A_j$ (first part of RHS)

$\forall j \in N(i) \setminus N^*(i^*)$ change in cost is

$$\begin{aligned} d(j, \hat{i}) - d(j, i) &\le d(j, \hat{i}^*) + d(\hat{i}^*, \hat{i}) - d(j, i) \\ &\le d(j, \hat{i}^*) + d(\hat{i}^*, i) - d(j, i) \\ &\le d(j, i^*) + d(j, \hat{i}^*) = 2\, OPT_j \end{aligned}$$ □

Now, sum over all pairs $\in \mathcal{P}$ ($i^* \in S^*$ appear once, $i \in Z \cup O \setminus S$ appear at most twice)

$$0 \le \underbrace{\sum_{i^* \in S^*} \sum_{j \in N^*(i^*)}}_{= \sum_{j \in V}} (OPT_j - A_j) + \underbrace{2 \sum_{i \in S} \sum_{j \in N(i)}}_{= \sum_{j \in V}} 2\, OPT_j = 5\, OPT - A$$

$$\Rightarrow \quad A \le 5\, OPT$$ □

Poly run time: Trick: only make changes if they improve at least a factor $1 - \delta$.

Then for each pair:

$$\delta \sum_{j \in V} A_j \le \sum_{j \in N^*(i^*)} (OPT_j - A_j) + \sum_{j \in N^*(i)} 2\, OPT_j$$

and

$$k\delta A \le 5\, OPT - A \quad \Rightarrow \quad A \le \frac{5}{1 - k\delta}\, OPT$$

$$\delta = \frac{\varepsilon}{k(1+\varepsilon)} \quad \Rightarrow \quad A \le (1+\varepsilon)\, 5\, OPT$$

Let smallest distance $= 1$ (by scaling) and $M = \sum_{i \in U} \sum_{j \in V} d(i, j)$

Then, since $(1 - \delta)^{1/\delta} \le \frac{1}{e}$, $\frac{1}{\delta} \ln M$ iterations suffice to go from a solution with value M to one with value ≤ 1.