Approximation Algorithms

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The Traveling salesperson problem (TSP)

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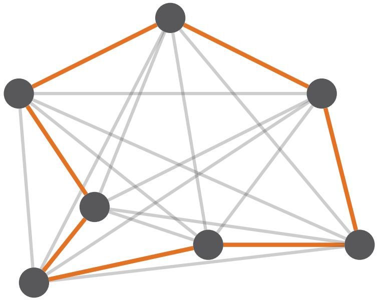
Input: complete graph $G = (V, E)$, distances $d : E \to \mathbb{R}_+$

Task: find a Hamiltonian cycle C in G minimizing $d(C) := \sum_{e \in C} d(e)$

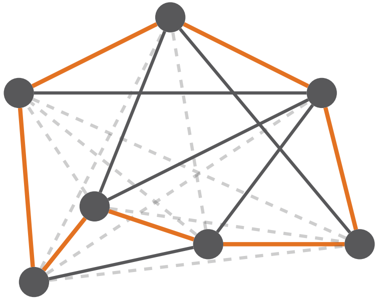


Hardness

Theorem

There is no α-approximation for TSP for any α, unless $P = NP$.

Proof Deciding whether graph has a Hamiltonian cycle is *NP*-hard.



Given graph $G' = (V, E')$ define complete graph $G = (V, E)$ with $d(e) = 0$ if $e \in E'$ and $d(e) = 1$ if $e \notin E'$.

$\textsc{Yes}$ instance: $\text{OPT} = 0$ $\textsc{No}$ instance: $\text{OPT} \geq 1$ $\square$

Metric TSP

Input: complete graph $G = (V, E)$, distances $d : E \to \mathbb{R}_+$, with $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V$

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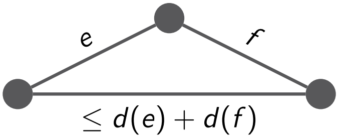
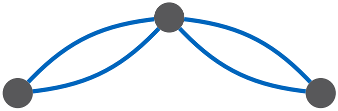


Shortcuts



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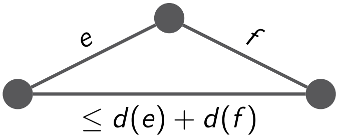

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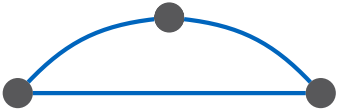


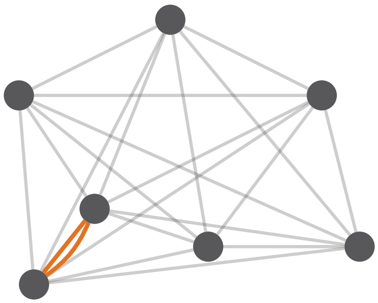
A greedy algorithm

Algorithm: Greedy

1. $C = 2 \times$ shortest edge, $\underset{\{u,v\} \in E}{\operatorname{argmin}} d(u, v)$.
2. $S = \{u, v\}$; $\bar{S} = V \setminus \{u, v\}$.

While $S \neq V$:

3. Find $u \in S$ and $v \in \bar{S}$, s.t. $d(u, v)$ is minimal.
4. Add v to S, and $\{u, v\}$ and $\{v, u'\}$ to C, where u' was u's successor.



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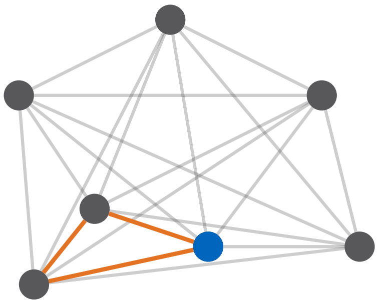

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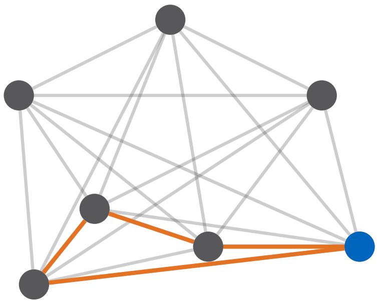

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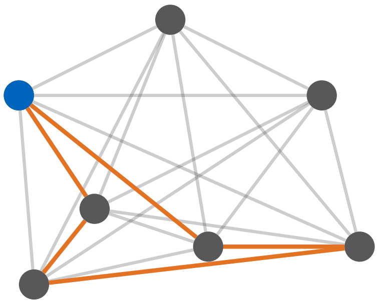

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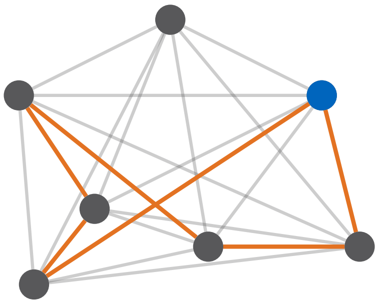

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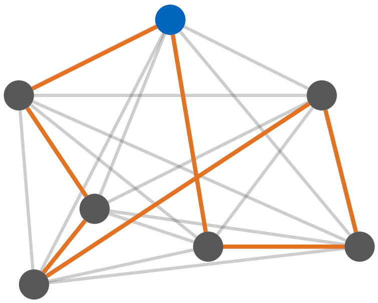
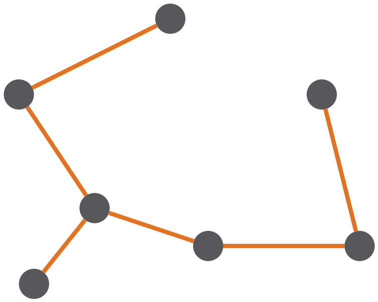


Minimum spanning tree

Minimum spanning tree (MST)

A subset $T \subseteq E$ is a tree if it

1. connects all vertices
2. contains no cycles
3. has minimum total length



Prim's algorithm:

1 $T = \left\{ \operatorname*{argmin}_{\{u,v\} \in E} d(u,v) \right\}$.

2 $S = \{u, v\}$; $\bar{S} = V \setminus \{u, v\}$.

While $S \neq V$:

3 $e = \operatorname*{argmin}_{u \in S, v \in \bar{S}} d(u,v)$.

4 $T = T + e$.

5 $S = S + v$.

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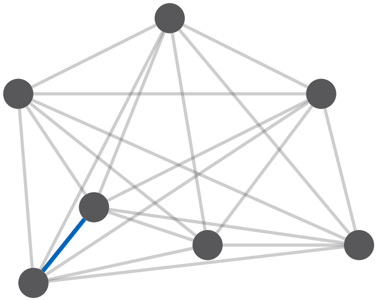
Theorem

Prim's algorithm computes a MST.



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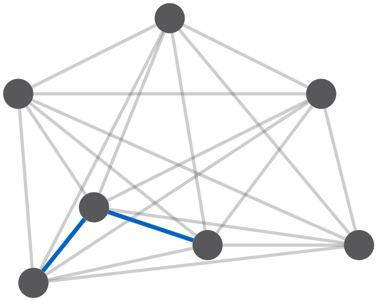

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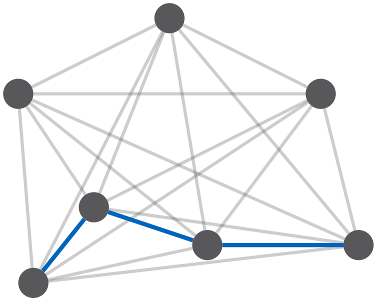

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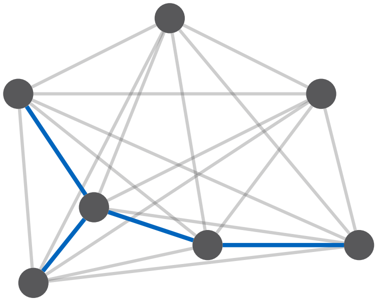

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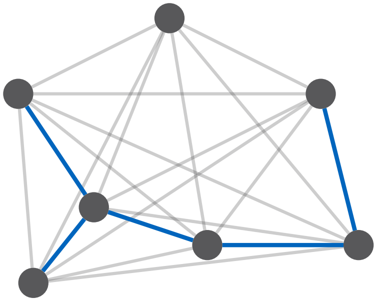

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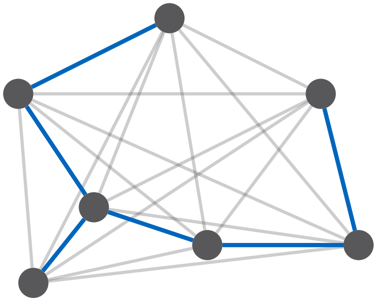


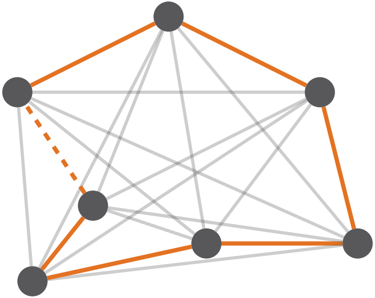
The tree lower bound

Lemma

Let C be a Hamiltonian cycle in G and T be a minimum spanning tree in G. Then $d(T) \leq d(C)$.

Proof. Let $e \in C$. Then $C \setminus \{e\}$ is a spanning tree in G. Hence

$$d(T) \leq d(C \setminus \{e\}) \leq d(C).$$



□

The Double-tree algorithm

Algorithm:

1. Compute MST T.
2. Let $H = (V,\ T \dot\cup T)$.
3. Compute Euler tour C' in H.
4. Shortcut C' to HC C.

Theorem

The Double-tree algorithm is a 2-approximation for metric TSP.

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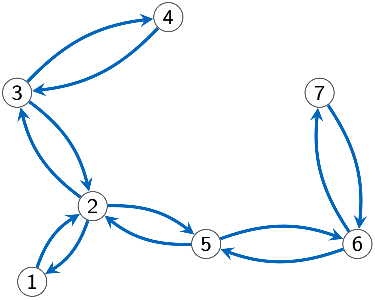
The Greedy algorithm for metric TSP is a 2-approximation.



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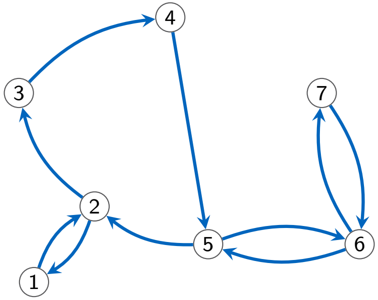

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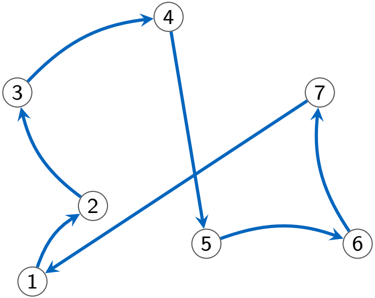


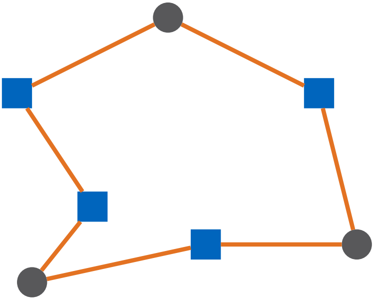
The Matching lower bound

Lemma

Let C be a HC in G and let $U \subseteq V$ with $|U|$ even. Let M be a minimum weight perfect matching on U. Then $d(M) \leq \frac{1}{2}d(C)$.

Proof. Shortcut C to cycle C' on U. $\qquad d(C') \leq d(C)$

C' contains two disjoint perfect matchings on U. $\qquad 2d(M) \leq d(C')$



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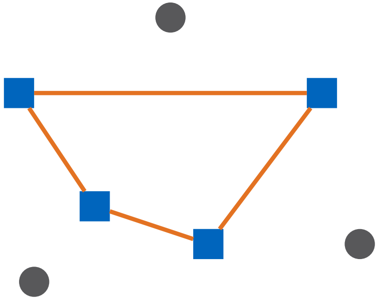

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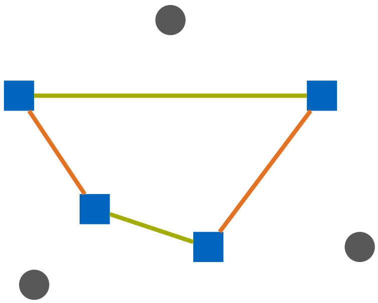

Christofides' algorithm

Algorithm:

1. Compute MST T.
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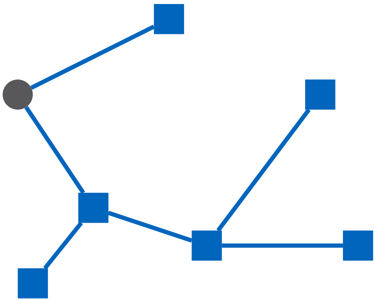

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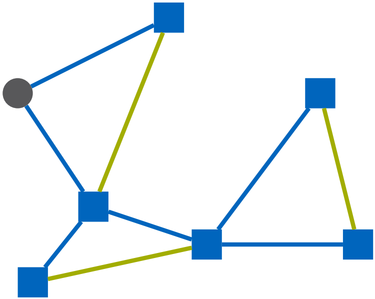

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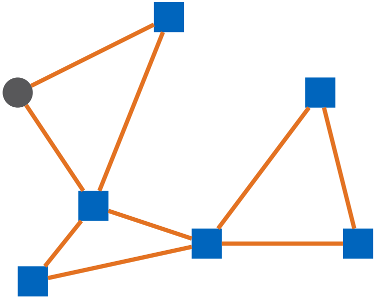

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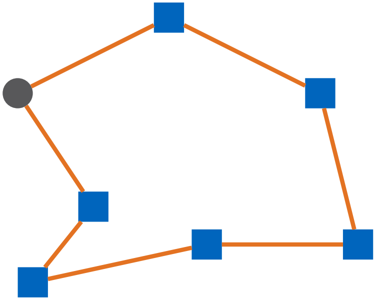

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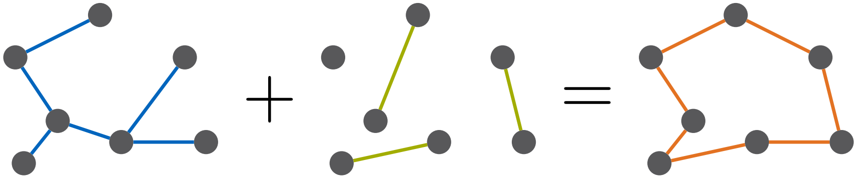

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