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Summer 2019

Approximation Algorithms

Homework Sheet 3 (Deadline 16. 05. 2019 12:00 – before the lecture starts.)

All exercises must be done individually. Feel free (and encouraged) to discuss among each otherut each solution must be written independently. Two or more submissions found with exactly the same solution on any of the exercises will be awarded no points for the entire exercise sheet. The same holds in case of directly copying from any other source.

Exercise 1 (*7 points*). Consider the problem of GRAPH BALANCING, where you get an undirected graph G with weights on the edges $p\colon E(G) \to \mathbb{R}^+$. Our goal is to make the graph directed so that the most loaded vertex (the vertex with the most weight directed towards it) has minimum possible load. Formally we seek an orientation of the edges which minimizes the goal function $u = \max_{v \in V} \sum_{e \in E; e \text{ directed to } v} p(e)$.

Formulate the problem as an integer linear program, create a linear programming relaxation of it and finally design and prove a 2-approximation algorithm for the problem.

Exercise 2 (*13 points*). We know from the lecture that the Christofides algorithm for the TRAVELLING SALESMAN PROBLEM satisfies ALG $\leq \frac{3}{2}$OPT, where ALG is the value of the solution for the algorithm and OPT is the value of the minimum/optimum solution.

We wish to prove (in several steps) that the Christofides algorithm is slightly stronger: namely, it is also true that ALG $\leq \frac{3}{2}\text{OPT}_{\text{LP}}$, where OPT_{LP} is the optimum value of the following linear relaxation:

$$\begin{aligned}
(P): \quad & \min \sum_{e \in E} c_e x_e \\
\forall v \in V: \quad & \sum_{e = vx} x_e = 2 \\
\forall S \subsetneq V, S \neq \emptyset: \quad & \sum_{e \in E(S, V \setminus S)} x_e \geq 2 \\
\forall e \in E: \quad & 0 \leq x_e \leq 1
\end{aligned}$$

The battle plan is as follows:

a) First verify that ALG $\leq \frac{3}{2}\text{OPT}_{\text{LP}}$ implies the original claim of ALG $\leq \frac{3}{2}$OPT.

b) Next, prove that for an optimum solution x^* of the LP (P) (that is precisely the point of value OPT_{LP}) it holds that $\frac{n-1}{n}x^*$ is a feasible solution for the spanning tree LP for the same graph.

c) Next, argue that Christofides' solution ALG can be compared to the optima of the spanning tree and matching LP below.

d) Finally, use points (b) and (c) to finish the claim that $\text{ALG} \le \frac{3}{2}\text{OPT}_{\text{LP}}$.

The *spannning tree LP* is the following program:

$$\min \sum_{e \in E} c_e x_e$$

$$\begin{aligned} & & \sum_{e \in E} x_e &= n - 1 \\ \forall S \subsetneq V, S \neq \emptyset\colon & & \sum_{e \in E(S, V \setminus S)} x_e &\ge 1 \\ \forall e \in E\colon & & x_e &\ge 0 \end{aligned}$$

The *minimum weight perfect matching LP* looks like this:

$$\min \sum_{e \in E} c_e x_e$$

$$\begin{aligned} \forall v \in V\colon & & \sum_{e = vx} x_e &= 1 \\ \forall S \subsetneq V, S \neq \emptyset, |S| \text{odd}\colon & & \sum_{e \in E(S, V \setminus S)} x_e &\ge 1 \\ \forall e \in E\colon & & x_e &\ge 0 \end{aligned}$$

You can make use of the following theorem, which was mentioned at the exercise session:

Theorem 1 (Integrality of the spanning tree and matching polytopes)**.** *The matching LPs and the spanning tree LPs are* integral *– the optimal LP solution and the optimal integer solution have the same value (assuming any integer solution exists). Furthermore, using an LP solver, one can compute the optimal integer solution in polynomial time.*