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Approximation Algorithms

Exercise Sheet 5 (28.05.2019)

Exercise 1 Consider the Knapsack problem

Definition 1 (Knapsack)**.**

Input: *a set of n items, N, with values v_i and sizes w_i, and a knapsack size B.*

Task: *find a subset $S \subset [n]$ such that $\sum_{i \in S} w_i \leq B$.*

Objective: *maximize $\sum_{i \in S} v_i$.*

a) Write an integer linear program that describes the Knapsack problem exactly.

Recall the that the integrality gap measures the performance of the LP relaxation compared to the ILP. The integrality gap of the ILP formulation of a problem is defined as the worst-case ratio over all instances of the problem of the value of an optimal solution to the ILP over the value of an optimal solution to the LP relaxation.

b) Prove that the integrality gap for the LP relaxation of your ILP is unbounded.

c) Suggest an improvement of the ILP by restricting the input. The restriction should be without loss of generality. What is the integrality gap for the improved ILP (provide an upper and a lower bound)?

Now consider the Knapsack cover problem

Definition 2 (Knapsack cover)**.**

Input: *a set of n items, N, with costs c_i and sizes w_i, and a knapsack size B.*

Task: *find a subset $S \subset [n]$ such that $\sum_{i \in S} w_i \geq B$.*

Objective: *minimize $\sum_{i \in S} c_i$.*

d) Suggest an ILP for the Knapsack cover problem. What is the integrality gap?

e) Suppose we round all item sizes to w'_i such that $w'_i = \min\{w_i, B\}$. Does this change the set of feasible solutions? Does this improve the integrality gap? Prove your answer.

Consider the following set of valid inequalities for the Knapsack cover problem

$$\sum_{i \notin S} x_i w_i(S) \geq B(S) \qquad \forall S \subset N\,,$$

where $B(S) = B - \sum_{i \in S} w_i$ and $w_i(S) = \min\{w_i, B(S)\}$.

f) Prove that the integrality gap of the ILP that includes these inequalities is exactly equal to 2. I.e., show an upper bound and a lower bound.

Exercise 2 Consider the Maximum flow problem

Definition 3 (Maximum flow)**.**

Input: *Directed graph $D = \{V \cup \{s\} \cup \{t\}, A\}$ with capacities $c_a \in \mathbb{N}$ for all $a \in A$. Note that arcs in A are directed.*

Task: *find a flow from s to t satisfying that the flow into each node in V is equal to the flow out of that node, that the flow on each arc is at most the capacity of that arc, and that the flow out of s is the flow into t.*

Objective: *maximize the flow out of s.*

a) Give an LP describing the Maximum flow problem.

b) Give the dual to the LP. The dual is known as the Minimum cut LP.

Consider the LP relaxation of the Price collecting steiner tree problem

$$\begin{aligned}
\min \quad & \sum_{e \in E} d(e)x(e) + \sum_{v \in V} \pi(v)(1 - y(v)) && && (1)\\
\text{s.t.} \quad & \sum_{e \in \delta(S)} x(e) \geq y(v) && \forall\, S \subseteq V, \forall v \in S && (2)\\
& y(v) \leq 1 && \forall v \in V && (3)\\
& x(e) \geq 0 && \forall e \in E && (4)\\
& y(v) \geq 0 && \forall v \in V && (5)
\end{aligned}$$

c) Give a polynomial time separation algorithm for the inequalities (2).
HINT: write an LP that minimizes $\sum_{e \in \delta(S)} x(e)$ over all sets $S \subseteq V$ that contain v.
HINT 2: these sets S do not contain the root r.