Universität Bremen

FB3: Mathematik/Informatik

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Approximation Algorithms

Exercise Sheet 2 (25.04.2019)

Exercise 1 Consider the local search algorithm for the k-Median problem of last lecture.

The k-Median problem

Input: a set of points V and metric distances $d : V^2 \to \mathbb{N}$.
Task: find k facilities in V, i.e., find $S \subseteq V$ with $|S| \leq k$.
Objective: minimize $\sum_{i \in V} d(i, S)$ with $d(i, S) = \min_{s \in S} d(i, s)$

Algorithm 1 Local search for k-Median

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Initialize any set S of k facilities.
while ∃j ∈ V, ∃s ∈ S s.t. ∑_{i∈V} d(i, S + j − s) < ∑_{i∈V} d(i, S) do
    S ← S + j − s.
end while
```

a) Suppose that in each iteration of the algorithm the solution improves by exactly 1. Give an upper bound on the number of iterations of the algorithm in terms of the $d(i, j)$'s. Is this bound polynomial in the input?

b) Suppose that in each iteration of the algorithm the solution improves by at least a factor of $1 - \delta$ for some fixed $\delta > 0$. Prove that the number of iterations of the algorithm is polynomial in the input. You may use that $(1 - \delta)^{1/\delta} \leq \frac{1}{e}$.

Algorithm 2 $(1 - \delta)$-step local search for k-Median

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Initialize any set S of k facilities.
while ∃j ∈ V, ∃s ∈ S s.t. ∑_{i∈V} d(i, S + j − s) < (1 − δ) ∑_{i∈V} d(i, S) do
    S ← S + j − s.
end while
```

c) Prove that Algorithm 2 is a $\frac{5}{1-k\delta}$-approximation algorithm for the k-Median problem. HINT: First show that

$$\delta \sum_{j \in V} A_j \leq \sum_{j \in N^*(i^*)} (\mathrm{OPT}_j - A_j) + \sum_{j \in N(i)} 2\mathrm{OPT}_j \, .$$

Note that we can choose $\delta = \frac{\varepsilon}{k(1+\varepsilon)}$, such that the approximation factor is only worse by a factor of $(1 + \varepsilon)$, for some arbitrarily small $\varepsilon > 0$.

Exercise 2 Consider the Uncapacitated facility location problem.

The Uncapacitated facility location problem

Input: a set of clients C and a set of facility locations F with metric distances $d_{ij} \in \mathbb{N}$ for all $i, j \in F \cup C$ and facility opening costs $f_i \in \mathbb{N}$ for all $i \in F$.
Task: open a subset S of the facilities to connect all clients to.
Objective: minimize $V(S) = \sum_{i \in S} f_i + \sum_{j \in C} d(j, S)$ with $d(j, S) = \min_{i \in S} d_{ij}$.

Also, consider the following local search algorithm that starts with a subset S of facilities and allows three different local moves. Namely, removing one facility from S (a "delete" move), adding one facility to S (an "add" move), or swapping one facility between S and $F \setminus S$ (a "swap" move).

Algorithm 3 Local search for Uncapacitated facility location

Initialize $S \leftarrow F$.
while $\exists S' \subset F$, s.t. $|S \setminus S'| \leq 1$ and $|S' \setminus S| \leq 1$ and $V(S') \leq V(S)$ **do**
  $S \leftarrow S'$.
end while

In this exercise we will analyze the performance of this local search algorithm. We start by showing that the outcome of the algorithm is a 3-approximation. Then, we show that we can improve on that approximation guarantee and that we need a small change to get the algorithm to run in polynomial time.

We say that a solution S is locally optimal if none of the local moves give any improvement. The algorithm terminates when it has found a locally optimal solution. For a given solution S and optimal solution S^*, denote by D and D^* the total distance to connect the clients to the facilities in S and S^*, respectively. Similarly, denote by K and K^* the total opening cost for the facilities in S and S^*, respectively. Lastly we let $\sigma(j)$ and $\sigma^*(j)$ denote the facilities in S and S^*, respectively, that are closest to j.

Let S be a locally optimal solution and S^* an optimal solution.

a) Prove that
$$f_{i^*} \geq \sum_{j : \sigma^*(j) = i^*} (d_{\sigma(j)j} - d_{\sigma^*(j)j})$$
for all $i^* \in S^* \setminus S$.

b) Prove that
$$f_{i^*} \geq \sum_{j : \sigma^*(j) = i^*} (d_{\sigma(j)j} - d_{\sigma^*(j)j})$$
for all $i^* \in S^* \cap S$.

c) Prove that
$$D \leq K^* + D^* .$$

Let $\varphi : S^* \to S$ map facilities in S^* to the nearest facility in S.

d) Consider a client j for which $\sigma(j) = i$ is not equal to $i' = \varphi(\sigma^*(j))$. Prove that reassigning j from i to i' increases the total distance by at most $2d_{\sigma*(j)j}$.

We say a facility i is *safe* if, for each client j such that $\sigma(j) = i$, we have $\varphi(\sigma^*(j)) \neq i$. That is, each client j served by i in S, can be served by $\varphi(\sigma^*(j))$ if we would delete i from S.

e) Prove that, for all safe facilities $i \in S$, we have

$$-f_i + \sum_{j:\sigma(j)=i} 2d_{\sigma*(j)j} \geq 0\,.$$

f) Consider a facility i that is not safe (also, unsafe). Let $R_i^* = \{i^* \in S^* \mid \varphi(i^*) = i\}$ and let $i' \in R_i^*$ be the one closest to i. Derive an inequality that expresses that an add move that adds a facility $i^* \in R_i^* - i'$ and reassigns all clients j for which $\sigma(j) = i$ and $\sigma^*(j) = i^*$ does not improve the objective.

g) Now consider the swap move that deletes an unsafe facility i and adds the facility $i' \in R_i^*$ that is closest to i. We reassign all clients $j : \sigma(j) = i$ as follows:

- If $\sigma^*(j) \notin R_i^*$ assign j to $\varphi(\sigma^*(j))$.
- If $\sigma^*(j) \in R_i^*$ assign j to i'.

Express the change in cost for this swap as an inequality for the case where $i \neq i'$. What changes if $i \neq i'$?

h) Prove that for any unsafe facility i it holds that

$$-f_i + \sum_{i^* \in R_i^*} f_{i^*} + \sum_{j:\sigma(j)=i} 2d_{\sigma^*(j)j} \geq 0\,.$$

HINT: sum the inequalities of exercises f) and g).

i) Show that $K^* - K + 2D^* \geq 0$ and use that to prove that a locally optimal solution has objective value not greater than 3 times the objective value of an optimal solution.

Note that we showed something that is stronger than just a 3-approximation. Namely, we showed that $D + K \leq 3D^* + 2K^*$. Next, we use this to improve the guarantee.

j) Say, we scale all facility costs such that the new cost for facility $i \in F$ is $f_i' = f_i/\mu$. Argue that

$$D + K' = D + K/\mu \leq 3D^* + 2K^*/\mu\,.$$

k) Improve upon your argumentation of the previous exercise to obtain

$$D + K \leq (1 + 2\mu)D^* + (1 + 1/\mu)K^*$$

and conclude that for suitably chosen μ, a locally optimal solution for the scaled local search algorithm has objective value not greater than $1 + \sqrt{2}$ times the objective value of an optimal solution for the non-scaled problem.

l) Use a similar reasoning as in Exercise 1 to show that we obtain a polynomial run time when we restrict the local moves to improve by at least $(1-\delta)$ factor in every iteration of the algorithm. Show that the resulting approximation guarantee is only a factor $(1+\varepsilon)$ worse than that of Algorithm 3. Use that $\frac{1}{1-\varepsilon/2} \leq 1 + \varepsilon$.