Hands-on Tutorial on Optimization

F. Eberle, R. Hoeksma, and N. Megow

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Modeling tricks

Linear programming tricks

$$
\begin{array}{llll}
\min & \displaystyle\sum_{j \in J} c_j x_j & & \\
\text{s.t.} & \displaystyle\sum_{j \in J} a_{ij} x_j \;\; \begin{matrix} \leq \\ = \\ \geq \end{matrix} \;\; b_i & & \forall i \\
& x_j \geq 0 & & \forall j
\end{array}
$$

Absolute values

$$\begin{array}{llll} \min & \sum_{j \in J} c_j |x_j| & & \\ \text{s.t.} & \sum_{j \in J} a_{ij} x_j \begin{array}{c} \leq \\ = \\ \geq \end{array} b_i & & \forall i \end{array}$$

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Introduce non-negative variables $x_j^+,\ x_j^-$.

$$\begin{aligned} x_j &= \\ |x_j| &= \\ x_j^+,\ x_j^- &\geq 0 \end{aligned}$$

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Introduce non-negative variables $x_j^+, \ x_j^-$.

$$\begin{aligned} x_j &= x_j^+ - x_j^- \\ |x_j| &= \\ x_j^+, \ x_j^- &\geq 0 \end{aligned}$$

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$$\begin{aligned} x_j &= x_j^+ - x_j^- \\ |x_j| &= x_j^+ + x_j^- \\ x_j^+, \ x_j^- &\geq 0 \end{aligned}$$

Absolute values

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j (x_j^+ + x_j^-) & & \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} (x_j^+ - x_j^-) \begin{array}{c} \leq \\ = \\ \geq \end{array} b_i & & \forall i \\ & x_j^+, \ x_j^- \geq 0 & & \forall j \end{aligned}$$

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$$\begin{aligned} x_j &= x_j^+ - x_j^- \\ |x_j| &= x_j^+ + x_j^- \\ x_j^+, \ x_j^- &\geq 0 \end{aligned}$$

Minimax objective

$$
\begin{array}{lll}
\min & \max\limits_{k \in K} \sum\limits_{j \in J} c_{kj} x_j & \\
\text{s.t.} & \sum\limits_{j \in J} a_{ij} x_j \; \begin{matrix} \leq \\ = \\ \geq \end{matrix} \; b_i & \forall i
\end{array}
$$

Minimax objective

$$\min \quad \max_{k \in K} \sum_{j \in J} c_{kj} x_j$$

$$\text{s.t.} \quad \sum_{j \in J} a_{ij} x_j \quad \begin{matrix} \leq \\ = \\ \geq \end{matrix} \quad b_i \qquad \forall i$$

Introduce variable z:

$$z = \max_{k \in K} \sum_{j \in J} c_{kj} x_j$$

$$\forall k \in K$$

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\end{array}
$$

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z = \max_{k \in K} \sum_{j \in J} c_{kj} x_j
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$$
\sum_{j \in J} c_{kj} x_j \leq z \qquad \forall k \in K
$$

Minimax objective

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\begin{aligned}
\min \quad & z & \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \begin{matrix} \le \\ = \\ \ge \end{matrix} b_i & \forall i \\
& \sum_{j \in J} c_{kj} x_j \le z & \forall k \in K
\end{aligned}
$$

Introduce variable z:

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z = \max_{k \in K} \sum_{j \in J} c_{kj} x_j
$$

$$
\sum_{j \in J} c_{kj} x_j \le z \qquad \forall k \in K
$$

Integer linear programming tricks

$$
\begin{array}{llll}
\min & \sum_{j \in J} c_j x_j & & \\
\text{s.t.} & \sum_{j \in J} a_{ij} x_j \overset{\le}{\underset{\ge}{=}} b_i & & \forall i \\
& x_j \in \{0,1\}/\mathbb{N} & & \forall j
\end{array}
$$

Jump in value

Constraint:

$$x = 0 \quad \text{or} \quad \ell \leq x \leq u$$

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Add binary variable y:

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } \ell \leq x \leq u \end{cases}$$

and constraints:

Jump in value

Constraint:

$$x = 0 \quad \text{or} \quad \ell \leq x \leq u$$

Add binary variable y:

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } \ell \leq x \leq u \end{cases}$$

and constraints:

$$\begin{aligned} x &\leq uy \\ x &\geq \ell y \\ y &\in \{0, 1\} \end{aligned}$$

Fixed costs

$$\begin{aligned} \min \quad & & C(x) & & \\ \text{s.t.} \quad & a_i x + \sum_{j \in J} a_{ij} w_j & \substack{\leq \\ = \\ \geq} & \; b_i & \forall i \\ & x & \geq & \; 0 & \\ & w_j & \geq & \; 0 & \forall j \end{aligned}$$

with

$$C(x) = \begin{cases} 0 & \text{for } x = 0 \\ k + cx & \text{for } x > 0 \end{cases}$$

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y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } \ell \le x \le u \end{cases}
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& x & \geq 0 & \\
& w_j & \geq 0 & \forall j
\end{array}
$$

$$
y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } \ell \leq x \leq u \end{cases}
$$

The cost function becomes:

$$
C^*(x, y) = ky + cx
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and we have constraints:

Fixed costs

$$\begin{aligned} \min \quad & C(x) \\ \text{s.t.} \quad & a_i x + \sum_{j \in J} a_{ij} w_j \lesseqqgtr b_i && \forall i \\ & x \geq 0 \\ & w_j \geq 0 && \forall j \end{aligned}$$

$$y = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } \ell \leq x \leq u \end{cases}$$

The cost function becomes:

$$C^*(x, y) = ky + cx$$

and we have constraints:

$$x \leq uy$$

$$y \in \{0, 1\}$$

Either or constraints

$$
\begin{aligned}
\min \quad & \sum_{j \in J} c_j x_j \\
\text{s.t.} \quad & \sum_{j \in J} a_{1j} x_j \leq b_1 \quad \text{or} \quad \sum_{j \in J} a_{2j} x_j \leq b_2 \\
& x_j \geq 0 \qquad \forall j
\end{aligned}
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y = \begin{cases} 0 & \text{if the first constraint holds} \\ 1 & \text{if the second constraint holds} \end{cases}
$$

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$$x_j \geq 0 \qquad \forall j$$

$$y = \begin{cases} 0 & \text{if the first constraint holds} \\ 1 & \text{if the second constraint holds} \end{cases}$$

$$\sum_{j \in J} a_{1j} x_j \leq b_1 + M_1 y$$

$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_2(1 - y)$$

$$y \in \{0, 1\}$$

If-then constraints

$$\begin{aligned} \text{If} \quad \sum_{j \in J} a_{1j} x_j &\leq b_1 \\ \text{then} \quad \sum_{j \in J} a_{2j} x_j &\leq b_2 \end{aligned}$$

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$$\sum_{j \in J} a_{1j} x_j > b_1 \quad \text{or} \quad \sum_{j \in J} a_{2j} x_j \leq b_2$$

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$$\sum_{j \in J} a_{1j} x_j \geq b_1 + \varepsilon \quad \text{or} \quad \sum_{j \in J} a_{2j} x_j \leq b_2$$

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$$\sum_{j \in J} a_{1j} x_j \leq b_1 - M_1 y$$
$$\sum_{j \in J} a_{2j} x_j \leq b_2 + M_2(1 - y)$$
$$y \in \{0, 1\}$$

Products of variables

Product $x_1 x_2$, with $x_1, x_2 \in \{0, 1\}$.

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Product $x_1 x_2$, with $x_1 \in \{0, 1\}$ and $0 \leq x_2 \leq u$.

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$$
\begin{aligned}
y &\leq u x_1 \\
y &\leq x_2 \\
y &\geq x_2 - u(1 - x_1) \\
y &\geq 0
\end{aligned}
$$

Special ordered sets (SOS1 and SOS2)

SOS1 and SOS2 are special types of variables.

SOS1 are positive variables that as a set can only have one non-zero.

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GAMS syntax:

```
SOS1 Variable s1(i);      SOS2 Variable s1(i);
```

Piecewise linear objective

Non-linear objective:

$$
\begin{aligned}
\min \quad & \sum_{j \in J} f_j(x_j) & & \\
\text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \;\begin{matrix}\leq\\=\\\geq\end{matrix}\; b_i & & \forall i \\
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Trick when $f(x)$ is separable, i.e. $f(x) = \sum_{j \in J} f_j(x_j)$.

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Trick when $f(x)$ is separable, i.e. $f(x) = \sum_{j\in J} f_j(x_j)$.

Example:

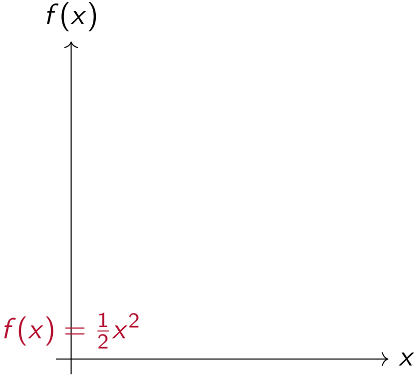
$$f(x) = x_1^2 + x_2 + e^{x_3}$$

Not separable:

$$f(x) = x_1 x_2 x_3$$

Piecewise linear objective

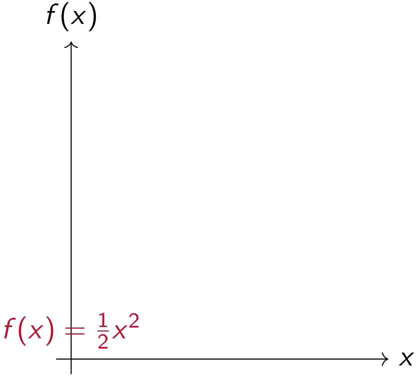
$$f(x) = \frac{1}{2}x^2$$



Piecewise linear objective

$$f(x) = \frac{1}{2}x^2$$

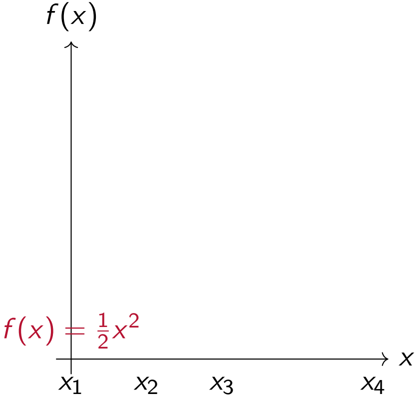
- Construct an approximation



Piecewise linear objective

$$f(x) = \frac{1}{2}x^2$$

- Construct an approximation
- Choose breakpoints x_1, x_2, x_3, x_4



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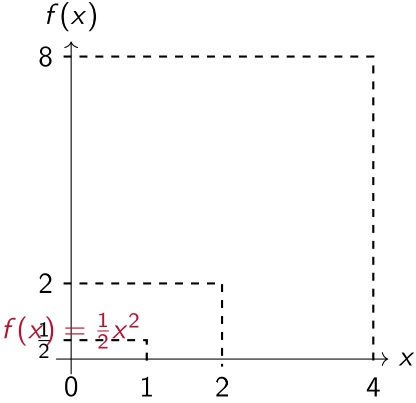

Piecewise linear objective

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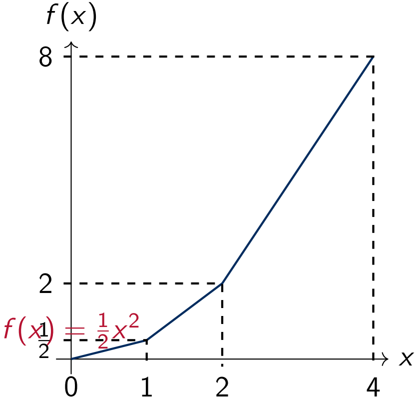
- Approximate the interval between x_i and x_{i+1} by a linear function



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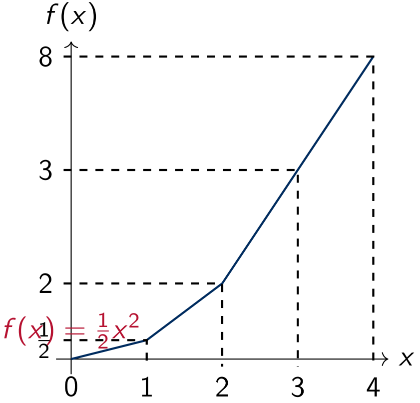
- Now $\tilde{f}(x) = \lambda_i f(x_i) + \lambda_{i+1} f(x_{i+1})$ for $x = \lambda x_i + (1 - \lambda) x_{i+1}$.



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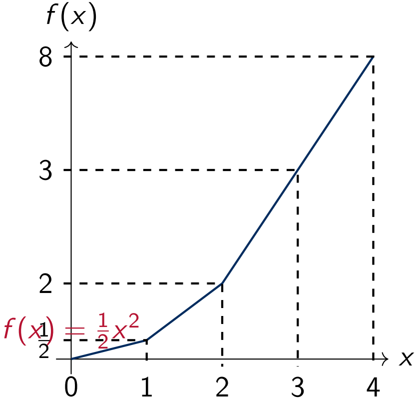
- Construct an approximation
- Choose breakpoints x_1, x_2, x_3, x_4
- Approximate the interval between x_i and x_{i+1} by a linear function
- Now $\tilde{f}(x) = \lambda_i f(x_i) + \lambda_{i+1} f(x_{i+1})$ for $x = \lambda x_i + (1-\lambda)x_{i+1}$.
- $3 = \frac{1}{2}2 + \frac{1}{2}4$, thus $\tilde{f}(3) = \frac{1}{2}f(2) + \frac{1}{2}f(4) = 1 + 4 = 5$



Piecewise linear objective

$$\tilde{f}(x) \;=\; \lambda_1 f(x_1) \;+\; \lambda_2 f(x_2) \;+\; \lambda_3 f(x_3) \;+\; \lambda_4 f(x_4)$$
$$x = \lambda_1 x_1 \;+\; \lambda_2 x_2 \;+\; \lambda_3 x_3 \;+\; \lambda_4 x_4$$
$$1 = \lambda_1 \;+\; \lambda_2 \;+\; \lambda_3 \;+\; \lambda_4$$

λ_i's are SOS2 variables.