Hands-on Tutorial on Optimization

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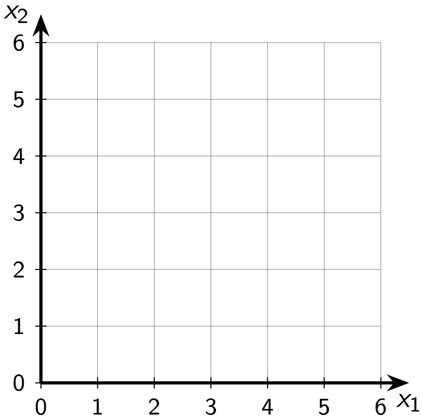
# Geometry of Linear Programs

# The Real, $n$-dimensional Space

$\mathbb{R}^n := \{x = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$

Elements $x \in \mathbb{R}^n$ can be seen as



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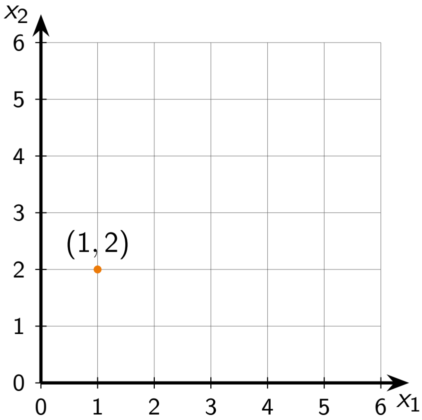

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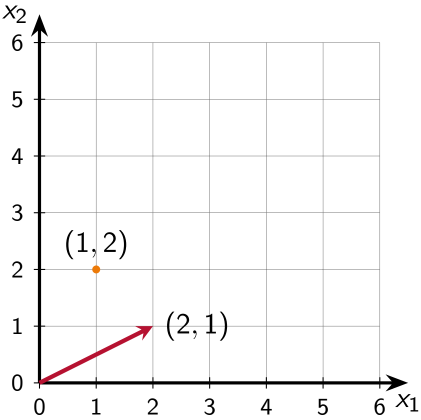

# The Real, $n$-dimensional Space

$\mathbb{R}^n := \{x = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$

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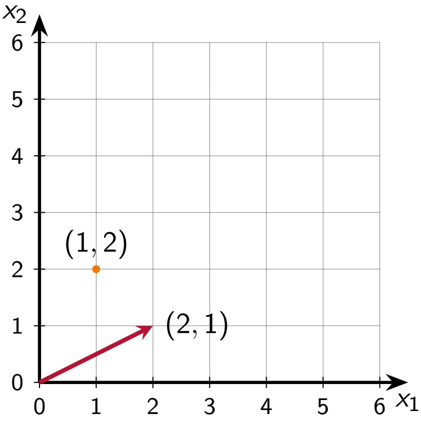
- Vectors

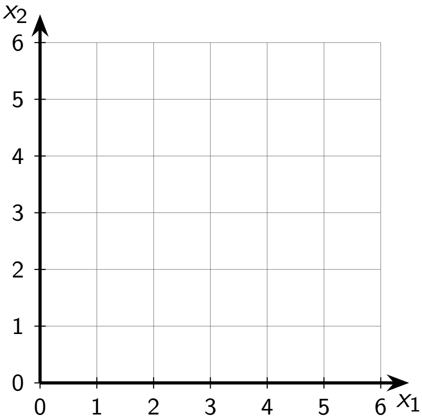
A $n$-tuple may represent the net profit of $n$ different goods, or their inventory level, or the cost of production, etc.



# Linear Equations

Let $a \in \mathbb{R}^n$ be a profit vector.

If you want the profit to be exactly $b$, how much should you produce?



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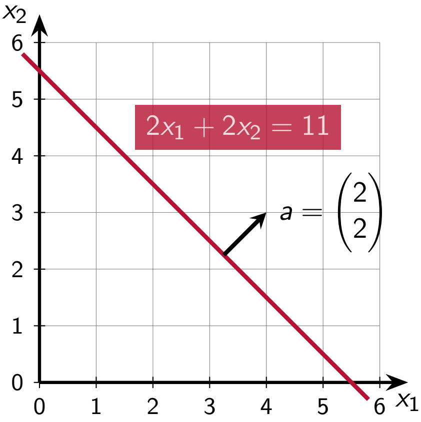
The answer is given as the set $\{x \in \mathbb{R}^n : a_1 x_1 + \ldots + a_n x_n = b\}$.

This is a line in $\mathbb{R}^n$.



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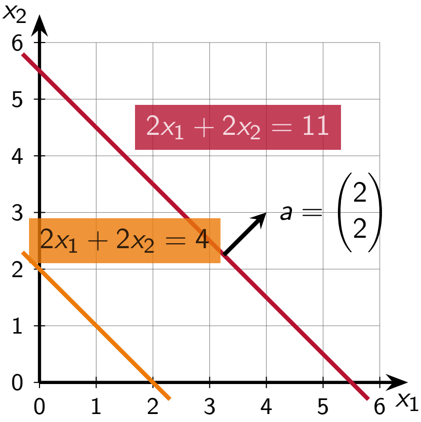
Changing the right hand side $b$ corresponds to "moving" the line along the vector $a$.



# Linear Inequalities

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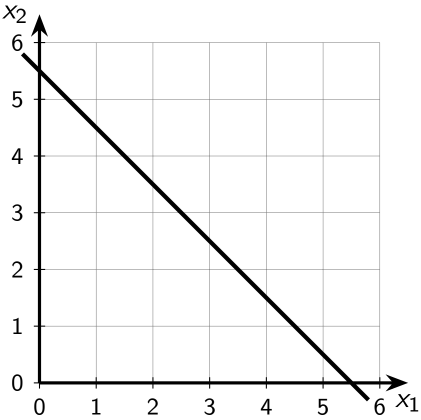

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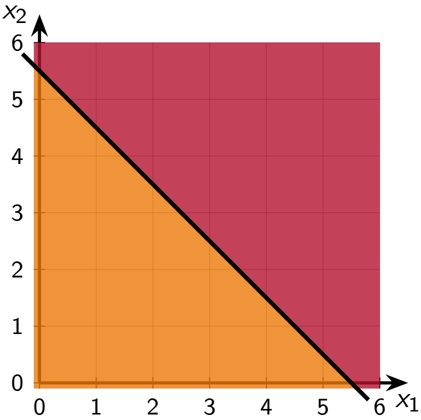
If you want to earn at least (at most) $b$, how much should you produce?

The answer is given as the set $\{x \in \mathbb{R}^n : a_1x_1 + \ldots + a_nx_n \geq (\leq) b\}$.

This is a halfspace in $\mathbb{R}^n$.



# Graphical Solution of LPs

Consider the chips factory problem.

$$
\begin{array}{lrcrcl}
\max & 2x_p & + & \frac{3}{2}x_m & & \\
\text{s.t.} & 2x_p & + & 4x_m & \leq & 345 \\
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\end{array}
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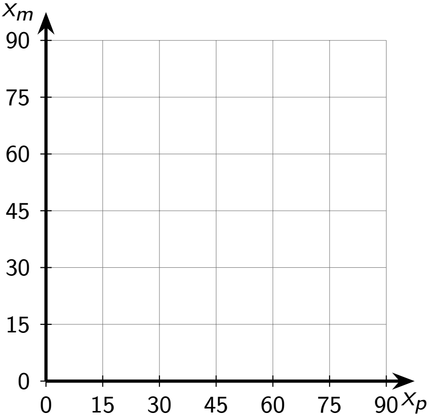

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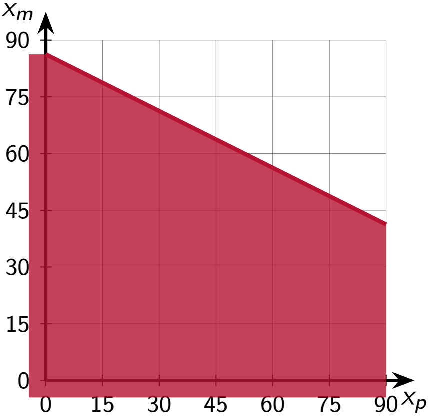

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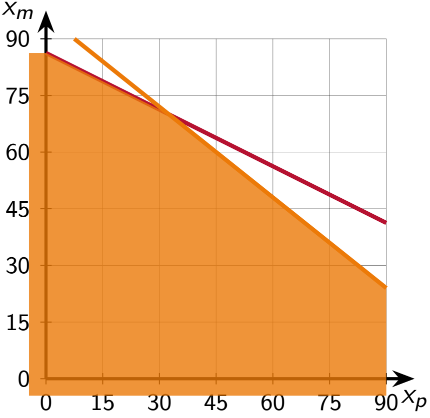

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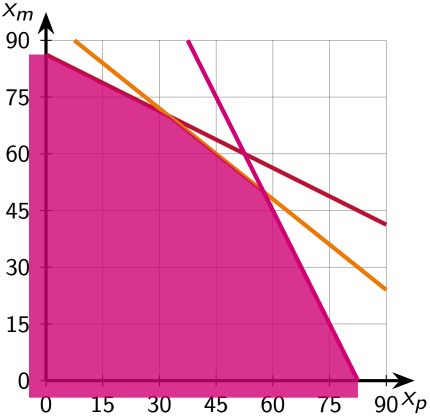

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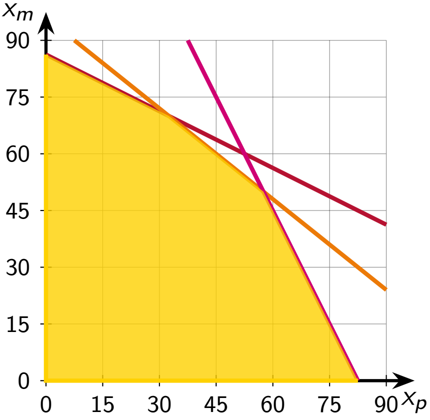

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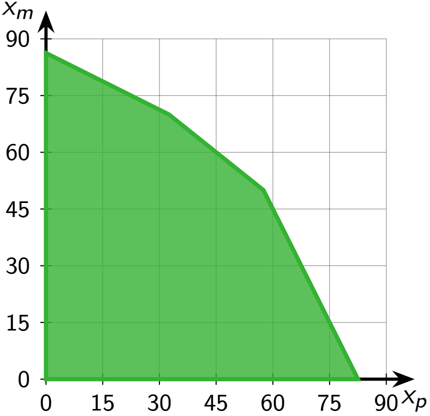

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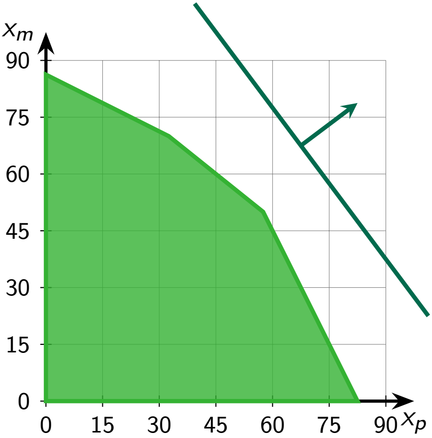

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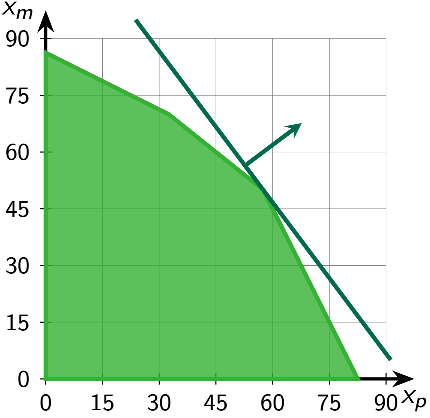

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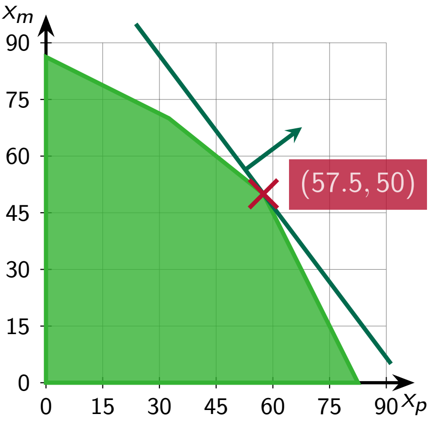
Optimal solution:
$(x_p, x_m) = (57.5, 50)$, s.t.
$2 \cdot 57.5 + \frac{3}{2} \cdot 50 = 190.$



# Graphical Solution of LPs

Consider the crude oil processing problem.
Processes:

| Process | cost per barrel | output per 10 barrels |
|---|---|---|
| 1 | 3€ | 2 barrels heavy oil |
| | | 2 barrels medium heavy oil |
| | | 1 barrel light oil |
| 2 | 5€ | 1 barrel heavy oil |
| | | 2 barrels medium heavy oil |
| | | 4 barrels light oil |

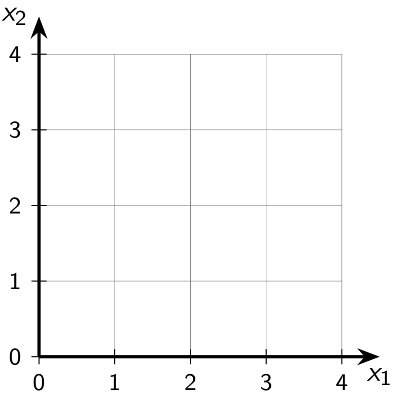
| | | |
|---|---|---|
| Demands: | heavy oil: | 3 barrels |
| | medium heavy oil: | 5 barrels |
| | light oil: | 4 barrels |

Construct the polyhedral representation and solve the problem then graphically by hand.

# Graphical Solution of LPs

Consider the crude oil processing problem.

$$\begin{array}{llcl} \min & 30x_1 + 50x_2 & & \\ \text{s.t.} & 2x_1 + 1x_2 & \geq & 3 \\ & 2x_1 + 2x_2 & \geq & 5 \\ & 1x_1 + 4x_2 & \geq & 4 \\ & x_1,\ x_2 & \geq & 0 \end{array}$$



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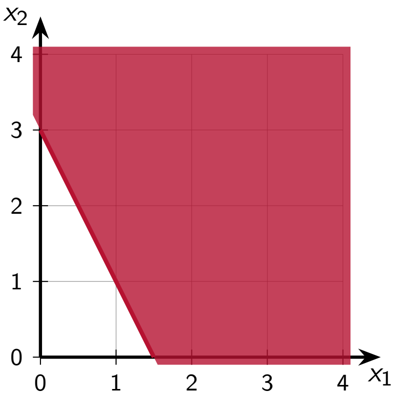

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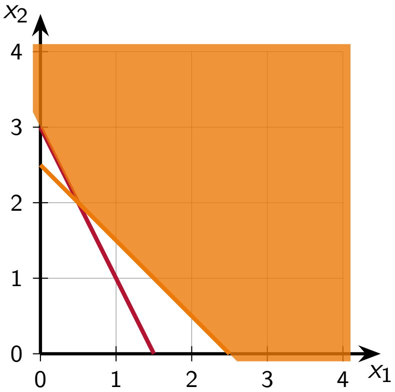

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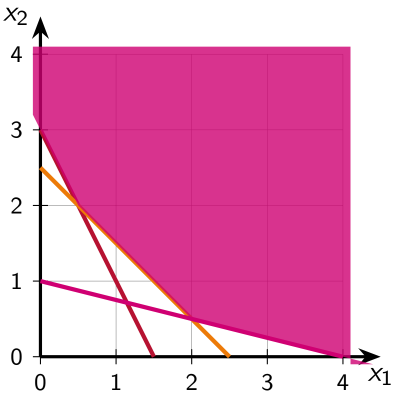

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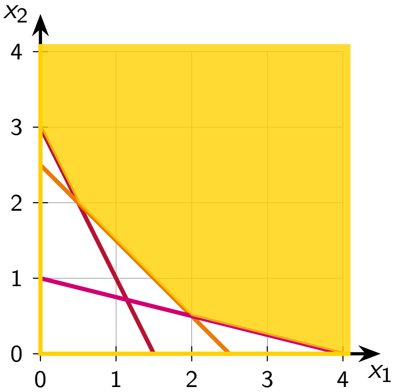

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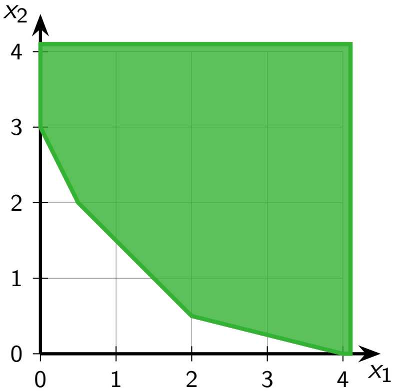

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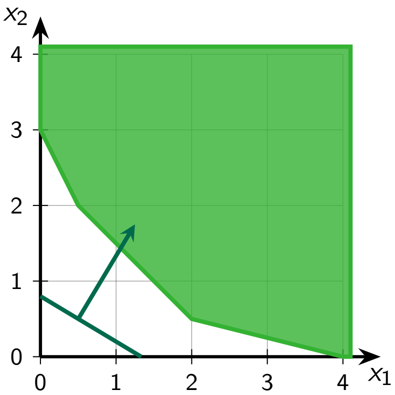

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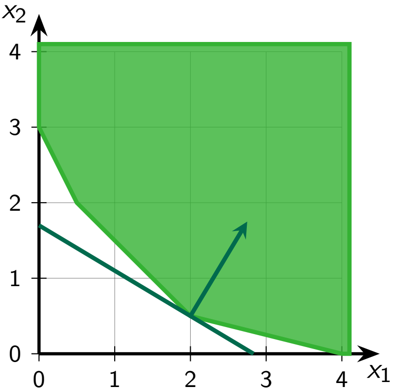

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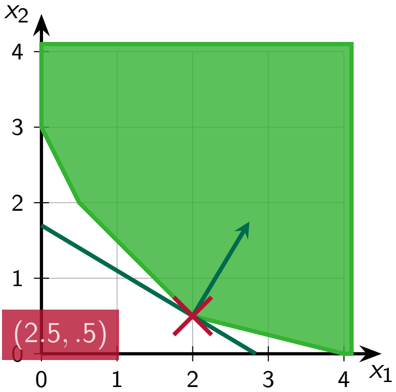


Optimal solution: $(x_1, x_2) = (2, .5)$,
s.t. $30 \cdot 2 + 50 \cdot .5 = 85$.

# Extreme points

**Definition**

Let $M \neq \emptyset$ convex. A point $x \in M$ is an extreme point of $M$ if we cannot find two points $y, z \in M \setminus \{x\}$ and a scalar $\lambda \in (0,1)$ such that

$$x = \lambda y + (1 - \lambda) z$$

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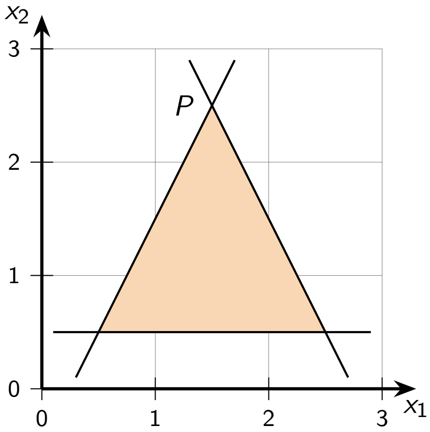
- Examples of convex sets with infinitely many extreme points: circle or ball

# Example

$$P = \left\{ x \in \mathbb{R}^2 : \begin{array}{rl} x_2 \geq & \frac{1}{2} \\ 2x_1 + x_2 \leq & \frac{11}{2} \\ -2x_1 + x_2 \leq & -\frac{1}{2} \end{array} \right\}$$



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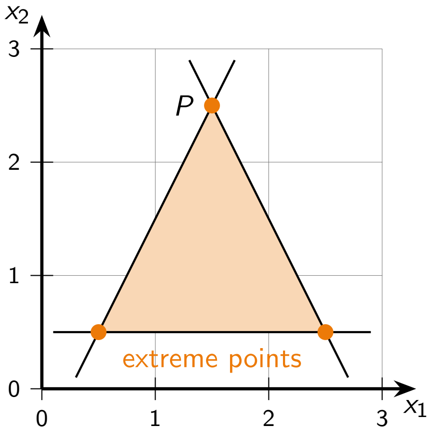


Break

# Matrices

A $m \times n$ dimensional matrix $A$ is an array of $n \cdot m$ real numbers $a_{i,j}$:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

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The transposed of $A$ is defined as follow:

$$A^T := \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

# Special Matrices

Let $\mathbf{0} \in \mathbb{R}^{m \times n}$ denote the matrix that only contains 0.

By $\mathbf{1} \in \mathbb{R}^{n \times n}$ we denote the matrix that only consists of 0s except the diagonal:

$$\mathbf{1} := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

# Multiplication

Let $x, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Then, $c^T x := \sum_{j=1}^n c_j \cdot x_j$, and

$$Ax = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \sum_{j=1}^n a_{1,j} \cdot x_j \\ \sum_{j=1}^n a_{2,j} \cdot x_j \\ \vdots \\ \sum_{j=1}^n a_{m,j} \cdot x_j \end{pmatrix} \in \ R^m$$

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$$Ax = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \sum_{j=1}^n a_{1,j} \cdot x_j \\ \sum_{j=1}^n a_{2,j} \cdot x_j \\ \vdots \\ \sum_{j=1}^n a_{m,j} \cdot x_j \end{pmatrix} \in \ R^m$$

Thus,

$$\begin{aligned} &\max c^T x \\ &\quad Ax \leq b \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max \sum_{j=1}^n c_j \cdot x_j \\ &\sum_{j=1}^n a_{1,j} \cdot x_j \leq b_1 \\ &\sum_{j=1}^n a_{2,j} \cdot x_j \leq b_2 \\ &\qquad \vdots \qquad\quad \vdots \quad \vdots \\ &\sum_{j=1}^n a_{m,j} \cdot x_j \leq b_3 \end{aligned}$$

# Example

$$P = \left\{ x \in \mathbb{R}^2 : \begin{array}{rcrcr} & & x_2 & \geq & \frac{1}{2} \\ 2x_1 & + & x_2 & \leq & \frac{11}{2} \\ -2x_1 & + & x_2 & \leq & -\frac{1}{2} \end{array} \right\}$$

$$= \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 0 & -1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} x \leq \begin{pmatrix} -\frac{1}{2} \\ \frac{11}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}$$

# Polytopes and Polyhedra

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- A bounded polyhedron is called polytop.

# Example

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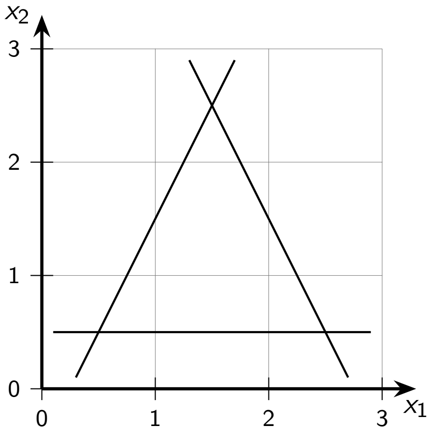

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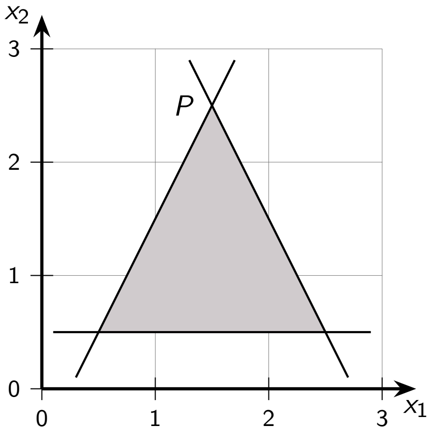

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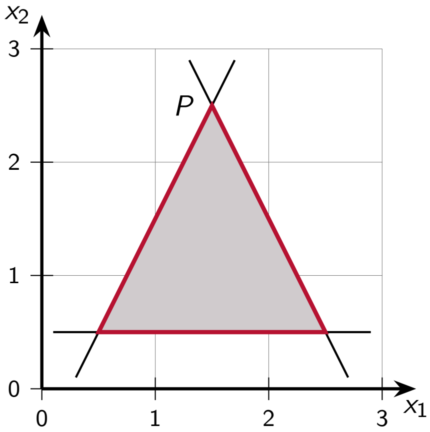

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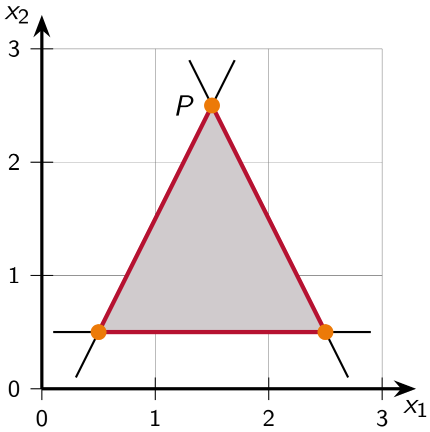


# Optimal Solutions

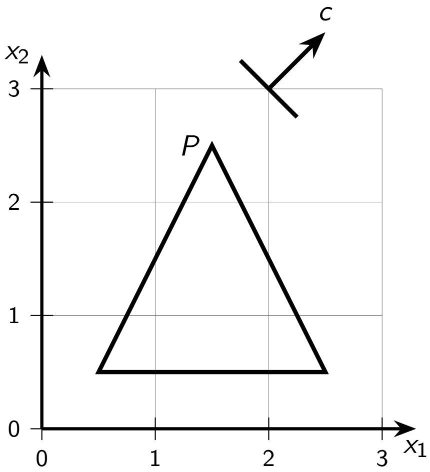
Let $P = P(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ a polyhedron.

**Theorem**

The optimum of $\min\{c^T x \mid x \in P\}$ is attained at a vertex of $P$.

# Example

$$\begin{array}{lrcrcr} \min & -x_1 & - & x_2 & & \\ \text{s.t.} & & & x_2 & \geq & \frac{1}{2} \\ & 2x_1 & + & x_2 & \leq & \frac{11}{2} \\ & -2x_1 & + & x_2 & \leq & -\frac{1}{2} \end{array}$$



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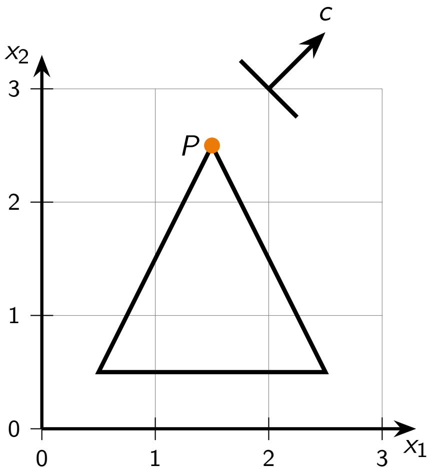
Optimal solution



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**Example:** Unit cube $P = \{x \in \mathbb{R}^n \,|\, 0 \leq x_i \leq 1, i = 1, \ldots, n\}$

Number of inequalities: $m = 2n$
Number of vertices: $2^n$

Works well in practice.