Hands-on Tutorial on Optimization

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Duality

Intuition

$$\begin{aligned} \min\ & 7x_1 + 3x_2 =: z(x) \\ \text{s.t.}\ & x_1 + x_2 \geq 2 \\ & 3x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

A feasible solution:
$x_1 = x_2 = 1$ with $z(x) = 10$.

Question: How to find lower bounds on the optimum?

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Idea: linear combination; consider coefficients $y_1 = 1$ und $y_2 = 2$.

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$$z(x) = 7x_1 + 3x_2 \geq 1 \cdot (x_1 + x_2) + 2 \cdot (3x_1 + x_2) \geq 1 \cdot 2 + 2 \cdot 4 = \mathbf{10}.$$

Hence, the above solution is optimal. ... now generalize.

Primal and dual program

Consider a general LP

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Obtain a lower bound

$$\begin{aligned} \max\ & b^T y \\ \text{s.t.}\quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

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Theorem

Let $\bar{x}$ be feasible a feasible solution for the primal LP (min) and let $\bar{y}$ be a feasible solution for the dual LP (max). Then

$$c^T \cdot \bar{x} \;\geq\; \bar{y}^T \cdot b.$$

Proof. ...

Examples

primal LP (min)	duales LP (max)
$\min\ c^T x$ s.t. $Ax \geq b$ $x \geq 0$	$\max\ b^T y$ s.t. $A^T y \leq c$ $y \geq 0$
$\min\ c^T x$ s.t. $Ax = b$ $x \geq 0$	$\max\ b^T y$ s.t. $A^T y \leq c$

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$\min\ c^T x$ s.t. $Ax = b$ $x \geq 0$	$\max\ b^T y$ s.t. $A^T y \leq c$

Theorem. The dual LP of a dual LP is the primal LP.

Dualization receipe

Primal linear program:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & a_i^T x \geq b_i && \text{for } i \in M_1 \\ & a_i^T x \leq b_i && \text{for } i \in M_2 \\ & a_i^T x = b_i && \text{for } i \in M_3 \\ & x_j \geq 0 && \text{for } j \in N_1 \\ & x_j \leq 0 && \text{for } j \in N_2 \\ & x_j \text{ free} && \text{for } j \in N_3 \end{aligned}$$

Dual linear program:

$$\begin{aligned} \max\ & y^T b \\ \text{s.t. } & y_i \geq 0 && \text{for } i \in M_1 \\ & y_i \leq 0 && \text{for } i \in M_2 \\ & y_i \text{ free} && \text{for } i \in M_3 \\ & y^T A_j \leq c_j && \text{for } j \in N_1 \\ & y^T A_j \geq c_j && \text{for } j \in N_2 \\ & y^T A_j = c_j && \text{for } j \in N_3 \end{aligned}$$

Primal & Dual Variables & Constraints

	primal LP (min)	dual LP (max)	
Constraints	$\geq b_i$	≥ 0	Variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
Variables	≥ 0	$\leq c_i$	Constraints
	≤ 0	$\geq c_i$	
	free	$= c_i$	

Strong Duality

Theorem

If the primal LP has an optimal solution, then there exists an optimal solution for the dual LP. The objective function values of both optimal solutions are equal.

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Possible primal-dual pairs:

primal\dual	optimal.	unbounded	infeasible
optimal	strong duality	impossible	impossible
unbounded	impossible	impossible	(1) weak duality
infeasible	impossible	(1) weak duality	(2) possible, c.f. Ex.

Examples

- Example for (1)

$$\begin{aligned} \min \quad & x_1 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \quad & | \; y_1 \\ & -x_1 - x_2 \geq 1 \quad & | \; y_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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- Example for (2)

$$\begin{aligned} \min\ & x_1 \\ \text{s.t.}\ & x_1 + x_2 \geq 1 \quad & \mid y_1 \\ & -x_1 - x_2 \geq 1 \quad & \mid y_2 \end{aligned}$$

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An application of strong duality

Interactive: Minimal Vertex Cover

Problem: Min Vertex Cover

Given: Graph $G = (V, E)$

Task: Find a minimal subset $V' \subseteq V$ such that each edge $e = \{u, v\} \in E$ has an endpoint in V'; we say e is covered.

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LP relaxation: Replace $x_v \in \{0, 1\}$ by $x_v \geq 0$.

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Dual LP for Minimal Vertex Cover

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$$\begin{aligned} \max \quad & \sum_{e \in E} y_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} y_e \leq 1, && \text{for all } v \in V \\ & y_e \geq 0, && \text{for all } e \in E. \end{aligned}$$

For $v \in V$ let $\delta(v) := \{e \in E \mid e = (u, v), u \in V\}$ the set of edges that are incident with v.

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Problem: Max Matching	
Given:	Graph $G = (V, E)$
Task:	Find a maximal matching, i.e., a maximal subset $M \subseteq E$ such that any vertex is incident to at most one edge $e \in M$.

Interactive: König's Theorem

Theorem

There are at least as many vertices in a minimal vertex cover as there are edges in a maximal matching.

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Definition. A Graph $G = (V, E)$ is bipartite, if there exists a partition $V = L \cup R$ such that there are no edges between L and R, i.e., there exists no $\{u, v\} \in E$ with $u \in L$ and $v \in R$.

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Theorem (König, 1931)

In a bipartite graph it holds that the number of vertices in a minimal vertex cover **equals** the number of edges in a maximal matching.

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Important: In general, weak and strong duality only hold for LPs (relaxations). However, it can be shown that the vertex cover and matching LPs always have an integral solution if the graph is bipartite.

Complementary slackness

Complementary Slack

Consider an arbitrary primal-dual pair (P) and (D).

Theorem

Let $\bar{x}$ be feasible for (P) and $\bar{y}$ feasible for (D). Then, $\bar{x}$ and $\bar{y}$ are optimal if and only if

$$\begin{aligned} \bar{x}_i \neq 0 \;\Rightarrow\; c_i = (\bar{y}^T A)_i \quad \text{for all } i, \\ \bar{y}_j \neq 0 \;\Rightarrow\; b_j = (A \cdot \bar{x})_j \quad \text{for all } j. \end{aligned}$$

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Observation. Either a variable vanishes (0) or the corresponding dual inequality has to be tight. A free variable corresponds to a equation in the dual that is tight by definition.