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Approximation Algorithms

Exercise Sheet 4

Exercise 1 (*10 points*) Consider the complete matching polytope as discussed in the lectures. Given a bipartite graph $G(A, B, E)$, we want to find a subset $M \subseteq E$ such that any vertex in A has exactly one endpoint in M and any vertex in B has at most one end point in B(assume $|A| \leq |B|$). The following is a natural feasibility LP-relaxation to this problem.

$$\sum_{i \in A} x_{ij} \leq 1, \forall j \in B$$
$$\sum_{i \in B} x_{ij} = 1, \forall j \in A$$
$$x \geq 0$$

Recall that an extreme point of a polytope is a feasible point x such that, it cannot be expressed as $\theta x^a + (1-\theta)x^b$, $0 \leq \theta \leq 1$, where x^a, x^b are two distinct feasible points in the polytope, other than x. Prove that any extreme point solution to the complete matching polytope is integral.

Exercise 2 (*10 points*) Let $x \in \mathbb{R}^n, y \in \mathbb{R}^m, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$. Consider the following primal-dual pair of linear programs.

$$\begin{aligned} \min c^T x \\ \text{subject to } Ax \geq b \\ x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max b^T y \\ \text{subject to } A^T y \leq c \\ y \geq 0 \end{aligned}$$

Let $x^\star, y^\star$ be a pair of feasible primal-dual solutions. Prove that if $x^\star, y^\star$ are optimal primal and dual solutions respectively, then they satisfy the following two conditions.

1. $x_i^\star > 0$ implies $\sum_j a_{ij} y_j^\star = c_i$ for all $i = 1, 2, \cdots n$.
2. $y_j^\star > 0$ implies $\sum_i a_{ij} x_i^\star = b_j$ for all $j = 1, 2, \cdots m$.

The above two conditions are known as **Complementary Slackness Conditions**.
(*Hint*: Use strong duality theorem)

Exercise 3 (*10 points*) Consider the following linear program.

$$\begin{aligned} \text{maximize } & 2x_1 + 4x_2 + 3x_3 + x_4 \\ \text{subject to } & 3x_1 + x_2 + x_3 + x_4 \leq 12 \\ & x_1 - 3x_2 + 2x_3 + 3x_4 \leq 7 \\ & x_1 + x_2 + 3x_3 - x_4 \leq 10 \\ & x \geq 0 \end{aligned}$$

a) (*3 points*) Write the dual of the above LP.

b) (*7 points*) $(0, 10.4, 0, 0.4)$ forms an optimal solution to the above LP. Use this fact and the complementary slackness conditions you proved in the previous exercises to determine an optimal dual solution.